Collection of formulae for Euler's constant γ

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1 Integral formulae

Euler's constant γ appears in many integrals (often related, for example, to the *gamma function* or the *logarithmic integral function*), we propose here to enumerate a selection of such integrals. Some of those can be deduced from others by elementary changes of variable.

We use the notation $\lfloor x \rfloor$ for the floor function and $\{x\}$ for the fractional part of a real number x.

$$
\begin{aligned}
1-\gamma &= \int_1^\infty \frac{t-\lfloor t \rfloor}{t^2} dt = \int_1^\infty \frac{\{t\}}{t^2} dt \\
-\gamma &= \int_0^\infty e^{-t} \log t dt = \Gamma'(1) \\
\gamma^2 + \frac{\pi^2}{6} &= \int_0^\infty e^{-t} \log^2 t \, dt = \Gamma^{(2)}(1) \qquad \text{(Euler-Mascheroni)} \\
-\gamma^3 - \frac{\gamma\pi^2}{2} - 2\zeta(3) &= \int_0^\infty e^{-t} \log^3 t \, dt = \Gamma^{(3)}(1) \qquad \text{(Euler-Mascheroni)} \\
\gamma &= -\int_0^1 \log\log \frac{1}{t} dt \\
\gamma + 2\log 2 &= -4\,\pi^{-1/2} \int_0^\infty e^{-t^2} \log t \, dt \\
\gamma &= \int_0^\infty e^{-t} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) dt \\
\log 2\pi - \gamma - \frac{1}{2} &= \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} \right)^2 dt \qquad ([4]) \\
\gamma &= \int_0^1 \left(\frac{1}{t} + \frac{1}{\log(1-t)} \right) dt \\
\gamma &= \int_0^\infty \left(\frac{1}{1+t} - e^{-t} \right) \frac{dt}{t} \\
\gamma &= \int_0^\infty \left(\frac{1}{1+t^2} - \cos t \right) \frac{dt}{t}
\end{aligned}
$$

$$
\begin{aligned}
\gamma &= \int_{-\infty}^{\infty} \frac{\log(1+e^{-t})e^t}{t^2+\pi^2} dt \qquad \text{(Prévost [11])} \\
(\alpha-\beta)\gamma &= \alpha\beta \int_0^{\infty} \frac{e^{-t^\alpha}-e^{-t^\beta}}{t} dt \qquad \alpha>0, \beta>0 \\
\gamma &= -\int_0^1 \int_0^1 \frac{1-x}{(1-xy)\log(xy)} dxdy \qquad \text{(Sondow [15])} \\
\gamma &= 1-\int_0^1 \frac{1}{1+t}\left(\sum_{k=1}^{\infty} t^{2^k}\right) dt \qquad \text{(Catalan)} \\
\gamma &= 1-\int_0^1 \frac{1+2t}{1+t+t^2}\left(\sum_{k=1}^{\infty} t^{3^k}\right) dt \qquad \text{(Ramanujan [13])} \\
\gamma &= \frac{1}{2}+2\int_0^{\infty} \frac{t\,dt}{(t^2+1)(e^{2\pi t}-1)} \qquad \text{(Hermite)} \\
\gamma &= 1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{2n}-\log n+\int_0^{\infty} \frac{2t\,dt}{(t^2+n^2)(e^{2\pi t}-1)} \\
\gamma &= \int_0^1 \frac{1-e^{-t}-e^{-1/t}}{t}\,\mathrm{d}\,t \qquad \text{(Barnes [1])} \\
\gamma &= \int_0^x \frac{1-\cos t}{t} dt - \int_x^{\infty} \frac{\cos t}{t} dt - \log x \qquad x>0 \\
\gamma &= \int_0^x \frac{1-e^{-t}}{t} dt - \int_x^{\infty} \frac{e^{-t}}{t} dt - \log x \qquad x>0
\end{aligned}
$$

This last integral is often used to deduce an efficient algorithm to compute many digits of γ (see [6]).

2 Series formulae

In this section we provide a list of various series for γ.

2.1 Basic series

$$
\begin{aligned}
\gamma &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) \qquad \text{(Euler)} \\
\gamma &= 1+\sum_{k\geq 2}\left(\frac{1}{k}+\log\left(1-\frac{1}{k}\right)\right) \qquad \text{(Euler)} \\
\gamma &= \log\frac{\pi}{4}+\sum_{k\geq 1}\left(\frac{1}{k}-2\log\frac{2k+2}{2k+1}\right) \\
\gamma &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\log\left(n(n+1)\right)\right) \qquad \text{(Cesaro)}
\end{aligned}
$$

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2}{2k-1}-\log(4n)\right)$$

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{2}\log\left(n^2+n+\frac{1}{3}\right)\right)$$

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{4}\log\left(\left(n^2+n+\frac{1}{3}\right)^2-\frac{1}{45}\right)\right)$$

$$\gamma = \lim_{n\to\infty}\left(2\left(1+\frac{(n-1)}{2n}+\frac{(n-1)(n-2)}{3n^2}+\cdots\right)-\log(2n)\right) \qquad \text{(Kruskal [9])}$$

$$\gamma = \lim_{s\to 1^+}\sum_{k\geq 1}\left(\frac{1}{k^s}-\frac{1}{s^k}\right) \qquad \text{(Sondow [14])}$$

$$\gamma = \lim_{n\to\infty}\left(n-\Gamma\left(\frac{1}{n}\right)\right) \qquad \text{(Demys)}$$

$$\gamma = \frac{\log 2}{2}+\frac{1}{\log 2}\sum_{k\geq 2}(-1)^k\frac{\log k}{k}$$

The last alternating series may be convenient to estimate Euler's constant to thousand decimal places thanks to convergence *acceleration of alternating series* (see the related essay at [6]).

2.1.1 Ramanujan's approach

In Ramanujan's famous *notebooks*, we find another kind of *Euler-Maclaurin* like asymptotic expansion; he writes

$$\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{2}\log\left(n(n+1)\right)\approx\gamma+\frac{1}{12p}-\frac{1}{120p^2}+\frac{1}{630p^3}-\frac{1}{1680p^4} \qquad (1)$$

with the variable $p=\frac{1}{2}n(n+1)$, which extends Cesaro's estimation. This representation may also be deduced from the classical Euler-Maclaurin expansion with Bernoulli's numbers.

2.2 Around the zeta function

When he studied γ, Euler found some interesting series which allow to compute it with the integral values of the Riemann zeta function. He used one of those to give the first estimation of his constant (a five correct digits approximation).

There are many formulae giving γ as function of the Riemann zeta function $\zeta(s)$, some are easy to prove. We provide the demonstration of one example.

By definition, we may write

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)=1+\sum_{k\geq 2}\left(\frac{1}{k}+\log\left(\frac{k-1}{k}\right)\right)$$

$$= \quad 1 + \sum_{k\geq 2}\left(\frac{1}{k} + \log\left(1 - \frac{1}{k}\right)\right)$$

and using the series for $\log(1-x)$ when $x = \frac{1}{k} < 1$ gives

$$\gamma = 1 - \sum_{k\geq 2}\left(\sum_{\ell\geq 2}\frac{1}{\ell k^\ell}\right)$$

then by associativity of this positive sum

$$\gamma = 1 - \sum_{\ell\geq 2}\frac{1}{\ell}\left(\sum_{k\geq 2}\frac{1}{k^\ell}\right) = 1 - \sum_{\ell\geq 2}\frac{1}{\ell}\left(\zeta(\ell) - 1\right).$$

So we have ve just demonstrated a first relation between γ and the zeta functions. Because it is clear that $\zeta(\ell) - 1$ is equivalent to $1/2^\ell$ when ℓ becomes large, some of those series have geometric convergence (of course one has to evaluate $\zeta(\ell)$ for different integral values of ℓ).

A general improvement can be made if we start the series with $k > 2$ by computing its first terms, that is, for any integer $n > 1$:

$$\gamma = 1 + \sum_{k=2}^{n}\left(\frac{1}{k} + \log\left(\frac{k-1}{k}\right)\right) + \sum_{k\geq n+1}\left(\frac{1}{k} + \log\left(\frac{k-1}{k}\right)\right)$$

and the result now becomes

$$\gamma = 1 + \sum_{k=2}^{n}\left(\frac{1}{k} + \log\left(\frac{k-1}{k}\right)\right) - \sum_{\ell\geq 2}\frac{1}{\ell}\left(\zeta(\ell) - 1 - \frac{1}{2^\ell} - \cdots - \frac{1}{n^\ell}\right),$$

and this time

$$\zeta(\ell, n+1) = \zeta(\ell) - 1 - \frac{1}{2^\ell} - \cdots - \frac{1}{n^\ell} \sim \frac{1}{(n+1)^\ell}$$

so that the rate of convergence is better. This function $\zeta(s,a)$ is known as the *Hurwitz Zeta function*. For different values of n, the identity for γ gives

$$\begin{aligned}
n &= 2 \qquad \gamma = \frac{3}{2} - \log 2 - \sum_{\ell\geq 2}\frac{1}{\ell}\left(\zeta(\ell) - 1 - \frac{1}{2^\ell}\right) \\
n &= 3 \qquad \gamma = \frac{11}{6} - \log 3 - \sum_{\ell\geq 2}\frac{1}{\ell}\left(\zeta(\ell) - 1 - \frac{1}{2^\ell} - \frac{1}{3^\ell}\right) \\
& \ldots
\end{aligned}$$

or in term of $\zeta(s,a)$ and the harmonic number H_n

$$\gamma = H_n - \log n - \sum_{\ell\geq 2}\frac{\zeta(\ell, n+1)}{\ell}. \tag{2}$$

2.2.1 Zeta series

$$\gamma = 1 - \sum_{k\geq 2} \frac{\zeta(k)-1}{k} \qquad \text{(Euler)}$$

$$\gamma = \sum_{k\geq 2} \frac{(k-1)(\zeta(k)-1)}{k} \qquad \text{(Euler)}$$

$$\gamma = 1 - \frac{\log 2}{2} - \sum_{k\geq 1} \frac{\zeta(2k+1)-1}{2k+1}$$

$$\gamma = \log 2 - \sum_{k\geq 1} \frac{\zeta(2k+1)-1}{k+1}$$

$$\gamma = 1 - \log\left(\frac{3}{2}\right) - \sum_{k\geq 1} \frac{\zeta(2k+1)-1}{4^k(2k+1)} \qquad \text{(Euler-Stieltjes)}$$

$$\gamma = 2 - 2\log 2 - \sum_{k\geq 1} \frac{\zeta(2k+1)-1}{(k+1)(2k+1)} \qquad \text{(Glaisher)}$$

$$\gamma = \sum_{k\geq 2} (-1)^k \frac{\zeta(k)}{k} \qquad \text{(Euler)}$$

$$\gamma = 1 - \log 2 + \sum_{k\geq 2} (-1)^k \frac{\zeta(k)-1}{k}$$

$$\gamma = \frac{3}{2} - \log 2 - \sum_{k\geq 2} (-1)^k (k-1) \frac{\zeta(k)-1}{k} \qquad \text{(Flajolet-Vardi)}$$

$$\gamma = \frac{5}{4} - \log 2 - \frac{1}{2}\sum_{k\geq 3} (-1)^k (k-2) \frac{\zeta(k)-1}{k}$$

$$\gamma = \log(8\pi) - 3 + 2\sum_{k\geq 2} (-1)^k \frac{\zeta(k)-1}{k+1}$$

$$\gamma = 1 + \log\left(\frac{16}{9\pi}\right) + 2\sum_{k\geq 2} (-1)^k \frac{\zeta(k)-1}{2^k k}$$

2.3 Other series

$$\gamma = \sum_{\ell\geq 1} \ell \sum_{k=2^\ell}^{2^{\ell+1}-1} \frac{(-1)^k}{k} \qquad \text{(Vacca [17], Franklin [5])}$$

$$\gamma = \log 2 - \sum_{\ell\geq 1} 2\ell \sum_{k=\frac{1}{2}(3^{\ell-1}+1)}^{\frac{1}{2}\left(3^\ell-1\right)} \frac{1}{(3k)^3 - 3k} \qquad \text{(Ramanujan [2])}$$

$$\gamma = \sum_{k\geq 1} (-1)^k \frac{\lfloor \log_2 k \rfloor}{k} \qquad \text{(Vacca [17])}$$

$$
\begin{aligned}
1-\gamma &= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{n}{k}\right\}\right) \qquad \text{(de la Vallée Poussin [18])}\\
\gamma &= \sum_{k\geq 1}\frac{a_k}{k} \qquad \text{(Kluyver)}\\
\gamma &= 1-\log 2+\sum_{k\geq 1}\frac{a_k}{k(k+1)} \qquad \text{(Kluyver)}\\
\gamma &= \sum_{k=1}^{n-1}\frac{1}{k}-\log n+(n-1)!\sum_{k\geq 1}\left(\frac{a_k}{k(k+1)\cdots(k+n-1)}\right) \qquad \text{(Kluyver [8])}
\end{aligned}
$$

In Kluyver's formulae the a_k are rational numbers defined by:

$$
a_1=\frac{1}{2}, \qquad a_k=\frac{1}{k+1}\sum_{\ell=1}^{k-1}\frac{k-\ell}{\ell(\ell+1)}a_{k-\ell}
$$

and $0<a_k\leq \frac{1}{k+1}$. Here are the first values:

$$
a_1=\frac{1}{2}, a_2=\frac{1}{12}, a_3=\frac{1}{24}, a_4=\frac{19}{720}, a_5=\frac{3}{160}, a_6=\frac{863}{60480}, a_7=\frac{275}{24192}.
$$

Kluyver's last relation may be used to compute a few thousand digits of γ.

3 Euler's constant and number theory

3.1 Dirichlet estimation

In 1838, Lejeune Dirichlet (1805-1859) showed that the mean of the *divisors function* $d(k)$ (numbers of divisors of k, [7]) of all integers from 1 to n is such as

$$
\frac{1}{n}\sum_{k=1}^{n}d(k)=\log n+2\gamma-1+O\left(\frac{1}{\sqrt{n}}\right).
$$

For example, a direct computation with $n=10^5$ produces

$$
\frac{1}{n}\sum_{k=1}^{n}d(k)-\log n=0.1545745350...
$$

while $2\gamma-1=0.1544313298....$

3.2 Mertens formulae

If p represents a prime number, Franz Mertens (1840-1927) gave in 1874 the two beautiful formulae ([10], [7]):

$$
e^{\gamma} = \lim_{n\to\infty}\frac{1}{\log n}\prod_{p\leq n}\left(1-\frac{1}{p}\right)^{-1} \tag{3}
$$

$$\frac{6e^\gamma}{\pi^2} = \lim_{n\to\infty} \frac{1}{\log n} \prod_{p\le n} \left(1+\frac{1}{p}\right) \tag{4}$$

The product (3) is equivalent to the series

$$\gamma = \lim_{n\to\infty} \left(\sum_{p\le n} -\log\left(1-\frac{1}{p}\right) - \log\log n \right) \tag{5}$$

but when p is large

$$-\log\left(1-\frac{1}{p}\right) = \frac{1}{p} + O\left(\frac{1}{p^2}\right)$$

and the relation (5) for γ is very similar to its definition relation, but this time, only the prime numbers are taken into account in the sum.

3.3 Von Mangoldt function

The *von Mangoldt function* $\Lambda(k)$ is generated by mean of the *Zeta function* as follow [7]:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{k\ge 1} \frac{\Lambda(k)}{k^s}, \qquad s>1 \tag{6}$$

and it is also defined by

$$\begin{cases} \Lambda(k) = \log p \;\text{ if } k = p^m \text{ for any prime } p, \\ \Lambda(k) = 0 \;\text{ otherwise.} \end{cases}$$

The relation (6) may also be written as

$$\zeta(s) + \frac{\zeta'(s)}{\zeta(s)} = -\sum_{k\ge 1} \frac{\Lambda(k)-1}{k^s}, \qquad s>1$$

from which, by taking the limits as s tends to 1, we deduce the interesting series expansion:

$$\gamma = -\frac{1}{2}\sum_{k\ge 1} \frac{\Lambda(k)-1}{k}. \tag{7}$$

It is a very slow and irregular converging series, partial sums S_n with n terms are

$$\begin{aligned} S_{1,000} &= 0.57(835...), \\ S_{10,000} &= 0.57(648...), \\ S_{100,000} &= 0.57(694...), \\ S_{1,000,000} &= 0.577(417...). \end{aligned}$$

4 Approximations

Unlike the constant π, few approximations are available for γ, it may be useful to list a few of those.

4.1 Rational approximations

The *continued fraction* representation makes it easy to find the sequence of the best rational approximations:

$$\gamma = [0; 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 1, 1, 40, 1, 11, 3, 7, 1, 7, 1, 1, 5, 1, 49, 4, 1, 65, ...],$$

that is, in term of fractions

$$\left[0, 1, \frac{1}{2}, \frac{3}{5}, \frac{4}{7}, \frac{11}{19}, \frac{15}{26}, \frac{71}{123}, \frac{228}{395}, \frac{3035}{5258}, \frac{15403}{26685}, \frac{18438}{31943}, \frac{33841}{58628}, \frac{289166}{500967}, \frac{323007}{559595}, ...\right].$$

For example, by mean of the continued fractions, we get the two approximative values

$$\left|\frac{33841}{58628} - \gamma\right| < 3.2 \times 10^{-11}$$

and

$$\left|\frac{376566901}{652385103} - \gamma\right| < 2.0 \times 10^{-19}.$$

A more exotic fraction due to Castellanos [3] is

$$\left|\frac{990^3 - 55^3 - 79^2 - 4^2}{70^5} - \gamma\right| < 3.8 \times 10^{-15}.$$

4.2 Other approximations

$$\begin{aligned}
\gamma &\approx \frac{1}{\sqrt{3}} = 0.577(350...) \\
\gamma &\approx \frac{41 - \sqrt{1241}}{10} = 0.57721(700...) \\
\gamma &\approx \frac{3}{43}\sqrt{66 + \sqrt{6}} = 0.577215(396...) \\
\gamma &\approx \frac{59}{3077}\left(1 + 11\sqrt{7}\right) = 0.577215664(894...)
\end{aligned}$$

$$
\begin{aligned}
\gamma &\approx \left(\frac{7}{83}\right)^{2/9} = 0.577215(209...) \qquad \text{(Castellanos [3])} \\
\gamma &\approx \left(\frac{80^3+92}{61^4}\right)^{1/6} = 0.577215664(572...) \qquad \text{(Castellanos [3])} \\
\gamma &\approx \frac{4}{2\sqrt{3}+5\log 2} = 0.57721(411...) \\
\gamma &\approx \frac{3}{3+2\log 3} = 0.5772(311...) \\
\gamma &\approx \frac{73}{293}\log\left(\frac{71}{7}\right) = 0.57721566(601...) \\
\gamma &\approx \frac{16}{241}+\log\left(\frac{5}{3}\right) = 0.57721566(525...) \\
\gamma &\approx \frac{3696}{43115}\log(840) = 0.5772156649015(627...)
\end{aligned}
$$

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