The Logarithmic Constant: $\log 2$

Xavier Gourdon and Pascal Sebah
numbers.computation.free.fr/Constants/constants.html

May 18, 2010

Abstract

An introduction to the history of the constant $\log 2$ and to its computation. It includes many formulas and some methods to estimate it, as well as the logarithm of any number, with few digits or to the greatest accuracy.

...by shortening the labors,
doubled the life of the astronomer.
- Pierre Simon de Laplace (1749-1827) about logarithms.

You have no idea, how much poetry there is
in the calculation of a table of logarithms!
- Karl Friedrich Gauss (1777-1855), to his students.

1 Introduction

1.1 Early history

The history of logarithms (*logos*=ratio + *arithmos*=number) really began with the Scot John Napier (1550-1617), of Merchiston Castle, in an opuscule published in 1614. This treatise was in Latin and entitled *Mirifici logarithmorum canonis descriptio* (*The Description of the Wonderful Canon of Logarithms)* [24, 25]. However, it should be pointed out that the Swiss clock-maker Jost Bürgi (1552-1632) independently invented logarithms but his work remained unpublished until 1620 [7, 8, 22].

Thanks to the possibility to replace painful multiplications and divisions by additions and subtractions respectively, this invention received an extraordinary welcome (in particular thanks to the enthusiasm of astronomers like Kepler) and spread rapidly on the Continent. George Gibson wrote during Napier Tercentenary Exhibition: “the invention of logarithms marks an epoch in the history of science” [13].

Soon, the need to modify Napier’s original work so that the logarithms of 1 and 10 become respectively 0 and 1 appeared. Under the impulsion of the

English mathematician and Napier's friend Henry Briggs (1561-1631) tables of *common* or *Briggian logarithms*, denoted $\log_{10}$ for logarithms having *base* 10, were computed. Briggs published his work in *Arithmetica logarithmica* during the year 1624 [6]; the tables were to an accuracy of fourteen digits and containing the common logarithm of all integers up to 20,000 and from 90,000 up to 101,000. The remaining gap was soon completed by the Dutchman Adrian Vlacq (1600-1667) in 1628 [35].

1.2 Natural logarithms

Mathematicians prefer to use the so-called *natural* or *hyperbolic logarithm* of a number (denoted log or ln, that is logarithms having *base* $e = 2.7182818284...$) and the following definition allows to derive easily the main properties of logarithms.

Definition 1 *Let $x > 0$, we set*

$$\log x = \int_1^x \frac{dt}{t} = \int_0^{x-1} \frac{dt}{1+t}. \tag{1}$$

Integral (1) may be interpreted as the area under the hyperbola $y = \frac{1}{t}$ with t going from 1 to x. This geometric interpretation was initiated in his *Opus geometricum* [28] published in 1647 by the Belgian Jesuit Grégoire de Saint-Vincent (1584-1667) and later completed by his student Alfonso Anton de Sarasa (1618-1667).

From this definition, the logarithm function is clearly a monotonous and continuous function. It is defined on the domain $]0, +\infty[$, positive for $x > 1$, negative for $0 < x < 1$ and $\log 1 = 0$. Its inverse is a positive function defined on the whole real domain, it is called the *exponential function* and is denoted $\exp x$ or e^x.

1.2.1 Modulus

Natural and common logarithms are simply linked by the relations

$$\log_{10} x = M \log x, \qquad \log x = \frac{1}{M} \log_{10} x, \tag{2}$$

where the proportion factor $M = \frac{1}{\log 10}$ is the *Modulus*: (It was computed in 1624 by Briggs to 16 correct decimal places [6] and by William Shanks in 1871 to more than 200 digits [32].)

$$M = 0.43429448190325182765112891891660508229439700580366...,$$
$$\frac{1}{M} = 2.30258509299404568401799145468436420760110148862877....$$

It should be observed that, in his historical work, Napier considered a logarithm (denoted Nlog) that can now be represented in term of natural logarithm

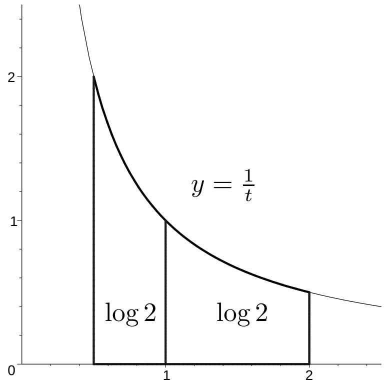

Figure 1: Two areas for $\log 2$

by:

$$\mathrm{Nlog}\, x = R \log \frac{R}{x} = -R \log x + R \log R, \qquad \text{with the radius } R = 10^7. \tag{3}$$

1.2.2 The constant $\log 2$

According to the integrals (1) we may now define the *logarithmic constant* or *Mercator's constant* $\log 2$.

Definition 2 *(Integral representation). The logarithmic constant is*

$$\log 2 = \int_1^2 \frac{dt}{t} = \int_0^1 \frac{dt}{1+t} = \int_0^{1/2} \frac{dt}{1-t}. \tag{4}$$

Finding the area under the hyperbola is very natural and we will see later that the constant also appears as the limit of one of the most elementary series. There is another domain where $\log 2$ is implicit: computer science. In most computer systems, all real numbers x are represented as binary floating-point:

$$x = m \times 2^E$$

where usually $1 \leq m < 2$ is the Mantissa and the integer E is the Exponent. Computing the logarithm of a real number may then be given by:

$$\log x = \log m + E \log 2$$

thus the computation of the logarithm of any number $x > 0$ is replaced by the computation of the logarithm of its mantissa m in the range $[1, 2[$.

Theorem 3 *(Weierstrass, 1885). The constant $\log 2$ is an irrational and transcendental number.*

Proof. Consult [36] where it is also established that $\log a$ is a transcendental number for any algebraic number $a \neq 1$ or [4] for a more modern approach. ■

Numerical estimations of its first decimal places and *regular continued fraction* are:

$$\log 2 = 0.693147180559945309417232121458176568075500134360255254120 6...,$$
$$\log 2 = [0; 1, 2, 3, 1, 6, 3, 1, 1, 2, 1, 1, 1, 1, 3, 10, 1, 1, 1, 2, 1, 1, 1, 1, 3, 2, 3, 1, 13, 7, ...].$$

A first value of the constant and a consequence of the relation (3) is implicitly given in Napier's original work [25]. He wrote: "All sines in the proportion of two to one have 6931469.22 for the difference of their logarithms", that is an error less than 3×10^{-7}. A few years later [18], Kepler found the *duplication logarithm* to be, with his convention, 69314.7193 so that the error is smaller than 2×10^{-8}.

2 Algorithms based on quadratures

It seems natural to start an estimation of the constant $\log 2$ from its integral representation (1). There are numerous tools to compute numerically definite integrals based on the idea to replace the function to integrate by an easier one. Usually piecewise polynomials of low degrees are used to achieve this. In this section we consider some classical quadratures and we deduce from them some elementary algorithms.

2.1 Rectangular quadrature

The simplest method is obtained from the classical *rectangular approximation* of the integral. The interval of integration $[a, b]$ is divided into smaller intervals on which the function is replaced by a constant approximation. In other words, let n be the number of subdivisions of the integration interval and let $h = \frac{1}{n}(b-a)$, then

$$\int_a^b f(t)\,dt = h\sum_{k=1}^{n} f(a+kh) + O(h).$$

Application of this rule to the function $f(t) = \frac{1}{1+t}$ on the domain $[0, 1]$ gives

$$\log 2 = \lim_{n\to\infty}\left(\frac{1}{n+1} + \cdots + \frac{1}{n+k} + \cdots + \frac{1}{n+n}\right). \qquad (5)$$

This is an increasing sequence bounded by 1 and its convergence is extremely slow with an error $O(\frac{1}{n})$. Computing d digits requires to evaluate the sum with $n \approx 10^d$ and therefore the convergence is *logarithmic*.

2.2 Trapezoid quadrature

The rectangular estimation can be improved if we replace on each small interval the function by its linear approximation. With the same definition for h, this produces the *trapezoidal rule* [20]

$$\int_a^b f(t)dt = \frac{h}{2}\left(f(a) + f(b) + 2\sum_{k=1}^{n-1} f(a+kh)\right) + O(h^2),$$

yielding

$$\log 2 = \lim_{n\to\infty}\left(\frac{3}{4n} + \left(\frac{1}{n+1} + \cdots + \frac{1}{n+k} + \cdots + \frac{1}{n+n-1}\right)\right). \tag{6}$$

The rate of convergence has been improved but it remains logarithmic and in $O(1/n^2)$.

2.3 Simpson's quadrature

Now let us consider a widely used improvement on the numerical computation of integrals and known as *Simpson's rule* [20]. The method is based on a piecewise parabola approximation:

$$\int_a^b f(t)dt = \frac{h}{6}\sum_{k=1}^{n}\left(f(a+kh) + f(a+(k-1)h) + 4f\left(a + \left(k - \frac{1}{2}\right)h\right)\right) + O(h^4).$$

It now produces

$$\log 2 = \frac{1}{6}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{n+k} + \frac{1}{n+k-1} + \frac{4}{n+k-\frac{1}{2}}\right) \tag{7}$$

with a new error in $O(1/n^4)$.

2.4 Numerical estimations

In the following table we compare the accuracy of the quadratures for different values of the number n of subdivisions of the interval.

n	Rectangular	Trapezoid	Simpson
10	0.6(687...)	0.693(771...)	0.693147(374...)
100	0.69(065...)	0.6931(534...)	0.6931471805(794...)
$1,000$	0.69(289...)	0.693147(243...)	0.69314718055994(726...)
$10,000$	0.6931(221...)	0.69314718(118...)	0.693147180559945309(612...)

From this simulation, we can notice that such elementary quadratures may only be used to compute a few digits of $\log 2$ or any other logarithm, the rate of convergence being too slow to calculate logarithms with significantly higher precision.

2.5 Other quadrature

Some more efficient numerical quadratures may in this particular case give some excellent results. One of those is the *Clenshaw-Curtis* quadrature [10] and it is based on the *Chebyshev's series expansion* of the function to integrate, that is the smooth function $t \to \frac{1}{t}$ in the $\log 2$ case. For such a regular function the convergence of the series expansion is much faster and geometric making the computation of thousand of digits now accessible. From this method, it can be stated after some manipulations that

$$\log 2 = \sqrt{2}\left(\frac{1}{2} - \sum_{k\geq 1} \frac{1}{(4k^2-1)\,(17+12\sqrt{2})^k}\right). \tag{8}$$

Partial sums s_n up to $k = n$ are given in the following table:

n	s_n
1	0.693(229...)
10	19 correct digits
100	159 correct digits
$1,000$	1539 correct digits

and this is a fast geometric convergence that can be obtained for any other logarithm.

3 Some simple limits

It is of interest to observe that logarithms may be computed by some very simple formulas that are not directly related to geometrical area and involving only square roots extractions.

3.1 Powering

There is a well known result that, for any real number z, the exponential function may be obtained by the classic limit

$$e^z = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n,$$

and this relation can be inverted to find:

$$\log x = \lim_{n\to\infty} n\left(x^{1/n} - 1\right) = \lim_{m\to\infty} 2^m \left(x^{1/2^m} - 1\right) = \lim_{\varepsilon\to 0} \frac{x^\varepsilon - 1}{\varepsilon}. \tag{9}$$

This limit is obviously deduced from the expansion

$$x^{1/n} = e^{\log x/n} = 1 + \frac{\log x}{n} + O\left(\frac{1}{n^2}\right)$$

and, for example, with $x = 2$ and $n = 2^{10} = 1024$, it produces the approximation 0.693(381...) with an error in $O(\frac{1}{n})$. A faster and more symmetric formula is given by

$$\log x = \lim_{n\to\infty} \frac{n}{2}(x^{1/n} - x^{-1/n}) = \lim_{m\to\infty} 2^{m-1}\left(x^{1/2^m} - x^{-1/2^m}\right) = \lim_{\varepsilon\to 0} \frac{x^\varepsilon - x^{-\varepsilon}}{2\varepsilon},$$

and again with $n = 1024$ we find 0.693147(233...), the error being now $O(1/n^2)$. This is certainly not the best way to compute a logarithm but it has the advantage to be extremely easy to apply to compute logarithms, with a low precision, by a few ($m = 10$ in our example) square roots extractions.

3.2 Infinite product

To conclude this section, we reproduce the unusual infinite product published by the German Ludwig von Seidel (1821-1896) [30] that is also a reformulation of (9).

Theorem 4 *(Seidel, 1871). Let the real number $x > 0$, then*

$$\frac{\log x}{x-1} = \prod_{k\geq 1} \frac{2}{1 + x^{1/2^k}} \tag{10}$$

and the partial products up to $k = n$ converge in $O(1/2^n)$.

Proof. For any real y, we know that

$$\frac{\sinh y}{y} = \frac{2}{y}\sinh\left(\frac{y}{2}\right)\cosh\left(\frac{y}{2}\right)$$

and repeating this to $\sinh\left(\frac{y}{2}\right)/\frac{y}{2}$ finally gives

$$y = \frac{\sinh y}{\cosh\left(\frac{y}{2}\right)\cosh\left(\frac{y}{2^2}\right)\cosh\left(\frac{y}{2^3}\right)\cdots}.$$

With $y = \log x$, the product becomes

$$\log x = \frac{x - x^{-1}}{2} \frac{2}{x^{1/2} + x^{-1/2}} \frac{2}{x^{1/4} + x^{-1/4}} \cdots$$

from which we deduce the equation (10). ■

If we apply formula (10) with $x = 2$, we find

$$\log 2 = \frac{2}{1+\sqrt{2}} \frac{2}{1+\sqrt{\sqrt{2}}} \frac{2}{1+\sqrt{\sqrt{\sqrt{2}}}} \cdots \tag{11}$$

and it can be compared with the similar product for the constant π and due to the French François Viète (1540-1603):

$$\frac{\pi}{2} = \frac{2}{\sqrt{2}} \frac{2}{\sqrt{2+\sqrt{2}}} \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdots.$$

4 Series expansions

4.1 Mercator's work

In 1668, the Danish Nicolas Mercator (1620-1687) published in his *Logarithmotechnia* [23] one of the first known series expansion of a function. It was established for the logarithmic function and opened the area of analytic calculations for logarithms.

Theorem 5 *(Mercator, 1668). Let x a real number and $-1 < x \leq 1$, then*

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k \geq 1} (-1)^{k-1} \frac{x^k}{k}.$$

Proof. Nowadays, we proceed by observing that for any integer n

$$\log(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x \left(1 - t + t^2 - \cdots + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t}\right) dt$$

so that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \int_0^x \frac{(-1)^n t^n}{1+t} dt.$$

To conclude, it is sufficient to show that the remainder

$$R_n = \left| \int_0^x \frac{(-1)^n t^n}{1+t} dt \right| \leq \left| \int_0^x \frac{t^n}{1+t} dt \right|,$$

tends to 0 as n tends to infinity. Clearly when $0 \leq x \leq 1$, we have

$$R_n \leq \int_0^x t^n dt = \frac{x^{n+1}}{n+1}$$

while for $-1 < x < 0$, we see that

$$R_n \leq \frac{1}{1+x} \left| \int_0^x t^n dt \right| = \frac{|x|^{n+1}}{(n+1)(1+x)}.$$

Hence in both cases, we conclude by observing that

$$\lim_{n \to \infty} R_n = 0.$$

■

For $|x| < 1$, the remainder of the series expansion tends to zero like $|x|^n$: this is a geometric convergence. Mercator illustrated the power of his method in computing $\log(1+0.1)$ (interpreted as the area under the hyperbola) to 43 correct digits.

Nevertheless, the direct application of Mercator's series for $x = 1$ is still valid and leads to the famous and very slowly convergent alternating series

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots . \tag{12}$$

It first values by taking $n = 10^m$ terms illustrate the speed of convergence:

n	s_n
10	0.6(456...)
100	0.6(881...)
$1,000$	0.69(264...)
$10,000$	0.693(097...)
$100,000$	0.69314(218...)

The *harmonic alternating series* (12) was studied by Pietro Mengoli of Bologna (1626-1686) and is remarkable for its simplicity and clearness.

The *Mercator series* or *logarithmic series* is comparable to another series studied by James Gregory (1638-1675) at the same time:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

producing with $x = 1$ the well known and very similar to (12) series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

4.2 Newton's computations

Isaac Newton (1643-1727) independently discovered Mercator's series and is among the first mathematician to take advantage of it for computational purposes. He made various estimations of logarithms in his celebrated *De methodis serierum et fluxionum* (*Method of Fluxions and Infinite Series*, [26]) written in 1671 but only published and translated from Latin to English in 1736, by John Colson (1680-1760).

He started with several accurate computations of $\log(1 + x)$ for small values of x like ± 0.1, ± 0.01, ± 0.2, ± 0.02, ... and was then able to compute logarithms of other numbers using the following relations

$$\begin{aligned} \log 2 &= 2\log(1 + 0.2) - \log(1 - 0.1) - \log(1 - 0.2), \qquad (13) \\ \log 3 &= \log 2 + \log(1 + 0.2) - \log(1 - 0.2), \\ \log 5 &= 2\log 2 - \log(1 - 0.2). \end{aligned}$$

By such observations, Newton then suggested an original and efficient method for building both a natural and a common accurate table of logarithms. He proposed to deduce the common logarithm table from the natural logarithm computations by mean of the *Modulus* and relations (2).

4.3 Easy approaches

An elementary idea is to apply Mercator's series with $x = -\frac{1}{2}$, this produces the very compact formula that was already known to Jacob Bernoulli (1654-1705):

$$\log 2 = \sum_{k \geq 1} \frac{1}{k2^k}. \tag{14}$$

The rate of convergence of this series is geometric (k iterations are needed to obtain one more digits) and consequently interesting to use for numerical hand or computer calculations. Some of it partial sums s_n are:

n	s_n
1	0.(5)
10	0.693(064...)
100	0.69314718055994530941723212145 81(688...)
$1,000$	304 correct digits

Relation (14) may also be deduced from the harmonic alternating series (12) by application of *Euler's transformation* on alternating series. (This is left as exercise.)

For a geometric series of ratio $\frac{1}{N}$, the number k of required terms to compute to obtain d correct decimal digits is given by the equation

$$\frac{1}{10^d} = \frac{1}{kN^k}.$$

It follows that for a large accuracy estimation this number is about

$$k \approx \frac{d}{\log_{10} N}. \tag{15}$$

and, in the case of series (14), we have $N = 2$ so that $k \approx 3.322d$.

Another alternative is to use the trivial decompositions

$$2 = \frac{3}{2}\frac{4}{3}, \qquad 2 = \left(\frac{4}{3}\right)^2 \frac{9}{8}$$

and take the logarithm of both side of those equalities to find the new similar formulas

$$\log 2 = \sum_{k \geq 1} \frac{1}{k}\left(\frac{1}{3^k} + \frac{1}{4^k}\right) \qquad \text{and} \qquad \log 2 = \sum_{k \geq 1} \frac{1}{k}\left(\frac{2}{4^k} + \frac{1}{9^k}\right).$$

4.4 BBP series

From the relation $\log 2 = -\frac{1}{2}\log(1 - \frac{1}{4}) + \operatorname{arctanh}\frac{1}{2}$, (See next paragraph for the definition of arctanh.) we deduce the relations

$$\log 2 = \sum_{k \geq 0} \left(\frac{1}{8k+8} + \frac{1}{4k+2}\right)\frac{1}{4^k}, \tag{16}$$

$$\log 2 = \frac{2}{3} + \sum_{k\geq 1}\left(\frac{4}{8k} + \frac{2}{8k+2} + \frac{1}{8k+4} + \frac{\frac{1}{2}}{8k+6}\right)\frac{1}{16^k}. \tag{17}$$

Like the ones discovered for π [3]:

$$\pi = \sum_{k\geq 0}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)\frac{1}{16^k},$$

those formulas can also be used to compute directly the n-th *binary* digit of $\log 2$ without computing the previous ones.

5 Machin like formulas

Using Mercator's series has improved a lot our ability to compute many digits for $\log 2$, but as for the constant π it is possible to go further. Before we need to recall the definition of the inverse hyperbolic tangent.

Definition 6 *(Gregory). Let $|x| < 1$ a real number, we define the function*

$$\operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = \sum_{k\geq 0}\frac{x^{2k+1}}{2k+1}.$$

5.1 Classical formulas

With the value $x = \frac{1}{3}$, this definition leads immediately to the interesting formula $\log 2 = 2 \operatorname{arctanh}\frac{1}{3}$ that is easy to use for large accuracy computations [19]. We expand it as:

$$\log 2 = \frac{2}{3}\sum_{k\geq 0}\frac{1}{(2k+1)9^k}, \tag{18}$$

and some of it partial sum s_n with the first n terms are:

n	s_n
1	0.6(666...)
10	0.6931471805(498...)
100	97 correct digits
1,000	957 correct digits

Now, it is natural to investigate if there exist more efficient series of this nature. There is a classical formula for π, known as *Machin's formula* and given by the celebrated relation:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

The idea is to search, by analogy, similar relations for $\log 2$. More precisely, let $(r_1, r_2, ..., r_n)$ be a set of rational numbers and $(d_1, d_2, ..., d_n)$ a set of increasing integers, we are looking for identities of the form

$$\log 2 = \sum_{k=1}^{n} r_k \operatorname{arctanh}\frac{1}{d_k}. \tag{19}$$

We also want the formula to be *efficient*. A good relation is a compromise between a few number of terms (n is small) and large values for the d_k. According to Lehmer [21] and to the relation (15), we can quantify this compromise and define the efficiency of an identity of the form (19).

Definition 7 *(Lehmer's measure). Let $d_k > 1$ a set of n integers, we define the efficiency E by*

$$E = E(d_1,\, d_2,\, ...,\, d_n) = \sum_{k=1}^{n} \frac{1}{\log_{10}(d_k^2)}.$$

For example the efficiency of $\log 2 = 2 \operatorname{arctanh} \frac{1}{3}$ is $E = 1.048$ (it corresponds to $n = 1$, $d_1 = 3$.) and, with the same definition, the efficiency of Machin's formula is $E = 0.926$. (it is obtained with $n = 2$, $d_1 = 5$, $d_2 = 239$.) Therefore, it will require a little more effort to compute $\log 2$ with formula (18) than to compute π with Machin's relation.

For any real number $k > 1$, it will be more convenient to introduce the notation:

$$\mathrm{L}(k) = \operatorname{arctanh} \frac{1}{k} = \frac{1}{2} \log \left(\frac{k+1}{k-1} \right), \tag{20}$$

so that the series expansion (18) becomes $\log 2 = 2\,\mathrm{L}(3)$.

It is easy to show the following decomposition theorem.

Theorem 8 *(Decomposition). Let $k > 1$ a real number, then*

$$\begin{aligned}
\mathrm{L}(k) &= \mathrm{L}(2k-1) + \mathrm{L}(2k+1),\\
\mathrm{L}(k) &= 2\,\mathrm{L}(2k-1) - \mathrm{L}(2k^2-1),\\
\mathrm{L}(k) &= 2\,\mathrm{L}(2k+1) + \mathrm{L}(2k^2-1),\\
\mathrm{L}(k) &= 2\,\mathrm{L}(2k) + \mathrm{L}(4k^3-3k).
\end{aligned}$$

Proof. Just replace $\mathrm{L}(k)$, $\mathrm{L}(2k-1)$, $\mathrm{L}(2k+1)$, ... by the definition (20) to check those identities. ∎

A direct application of this theorem for $k = 3$ gives new set of relations with $n = 2$.

Corollary 9 *We have the two terms identities:*

$$\log 2 = 2\,\mathrm{L}(5) + 2\,\mathrm{L}(7), \qquad \textit{Euler 1748} \tag{21}$$
$$\log 2 = 4\,\mathrm{L}(5) - 2\,\mathrm{L}(17), \tag{22}$$
$$\log 2 = 4\,\mathrm{L}(7) + 2\,\mathrm{L}(17), \tag{23}$$
$$\log 2 = 4\,\mathrm{L}(6) + 2\,\mathrm{L}(99) \tag{24}$$

with respective efficiencies $E = 1.307$, $E = 1.121$, $E = 0.998$ *and* $E = 0.893$.

Using other decomposition formulas and computer calculations based on the *LLL* algorithm [11], it is possible to find many other relations of this nature. (Most of them are given in [29].) To illustrate this, we have selected another formula with 3 terms and two others with 4 terms.

Theorem 10 *The constant* $\log 2$ *can be obtained by one of the following fast converging relations:*

$$\log 2 = 18\,\mathrm{L}(26) - 2\,\mathrm{L}(4801) + 8\,\mathrm{L}(8749), \tag{25}$$

$$\log 2 = 144\,\mathrm{L}(251) + 54\,\mathrm{L}(449) - 38\,\mathrm{L}(4801) + 62\,\mathrm{L}(8749), \tag{26}$$

$$\log 2 = 72\,\mathrm{L}(127) + 54\,\mathrm{L}(449) + 34\,\mathrm{L}(4801) - 10\,\mathrm{L}(8749) \tag{27}$$

with respective efficiencies $E = 0.616$, $E = 0.659$ *and* $E = 0.689$.

Proof. A posteriori, to establish formula (25) it is equivalent to check that

$$\left(\frac{27}{25}\right)^9 \left(\frac{4800}{4802}\right) \left(\frac{8750}{8748}\right)^4 = 2!$$

We proceed in the same way to verify all identities of this nature. ■

The formulas (26) and (27) were used to compute more than 10^8 digits for $\log 2$. One relation was used for the computation and the other for the verification. As we observe, the computation of only 5 different arctanh were necessary because of the similitude of those two formulas. Thanks to relation (25), the record was increased up to more than 6×10^8 a few years later and more than 3×10^{10} were achieved at the beginning of 2009.

A powerful method, based on the *binary splitting* process, to evaluate geometric series of the form $\mathrm{L}(k)$ is fully described in [15]. Notice that it is also possible to perform separately the computation of each term of the sum, making those approaches easy to compute in parallel.

The speed of convergence of the series expansion (26) can be appreciated if we compute its first terms:

n	s_n
1	0.69314(394...)
2	0.6931471805(304...)
3	0.69314718055994(497...)
4	0.69314718055994530941(317...)

and each iteration adds about $2\log_{10}(251) \approx 4.8$ digits.

5.2 Formulas with rational numbers

It may be convenient to look for relations with rational numbers as arguments of the arctanh function, that is identities of the form:

$$\log 2 = \sum_{k=1}^{n} a_k \ \mathrm{arctanh}\,\frac{n_k}{d_k},$$

with n_k being integers eventually different from 1. Lehmer's measure must be adapted to take this in account

$$E = \sum_{k=1}^{n} \frac{1}{2 \log_{10} \left(\frac{d_k}{n_k} \right)},$$

but it only gives a rough estimation of the real efficiency.

The following two relations are good candidates for accurate computations [29].

Theorem 11 *The new two identities*

$$\log 2 = 6\,\mathrm{L}(9) + 2\,\mathrm{L}\left(\frac{253}{3}\right), \tag{28}$$

$$\log 2 = 10\,\mathrm{L}(17) + 4\,\mathrm{L}\left(\frac{499}{13}\right) \tag{29}$$

have respective efficiencies $E = 0.784$ *and* $E = 0.722$.

Note that relation (29) was used for several high precision computations of the constant.

6 Variation with hypergeometric series

In his 1812 work, *Disquisitiones generales circa seriem infinitam*, the German mathematician, Carl Friedrich Gauss (1777-1855) studied the following family of series

$$F(a, b; c; z) = 1 + \frac{a.b}{c}\frac{z}{1!} + \frac{a(a+1).b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$$

where a, b, c are real numbers. The series converges in $|z| < 1$ and, on the unit circle $|z| = 1$, the series converges when $c > a + b$. Those functions are called *hypergeometric functions* and occur in many fields [1].

Many usual functions can be represented as hypergeometric functions with suitable values for a, b, c.

Example 12 *For any real number* x*, such as* $|x| < 1$*, we have the classical representations:*

$$\begin{aligned}
\log(1+x) &= x\,F(1,\, 1;\, 2;\, -x),\\
\operatorname{arctanh} x &= x\,F\left(\frac{1}{2},\, 1;\, \frac{3}{2};\, x^2\right),\\
\arctan x &= x\,F\left(\frac{1}{2},\, 1;\, \frac{3}{2};\, -x^2\right),\\
\frac{1}{1-x} &= F(1,\, 1;\, 1;\, x).
\end{aligned}$$

In 1797, Johann Friedrich Pfaff (1765-1825), a teacher and friend of Gauss, established the transformation

$$F(a,\,b;\,c;\,z) = (1-z)^{-b} F\left(c-a,\,b;\,c;\,\frac{z}{z-1}\right).$$

If we use this identity with the arctanh function

$$\operatorname{arctanh} x = x\,F\left(\frac{1}{2},\,1;\,\frac{3}{2};\,x^2\right),$$

we find

$$\operatorname{arctanh} x = \frac{x}{(1-x^2)} F\left(1,\,1;\,\frac{3}{2};\,\frac{x^2}{x^2-1}\right),$$

giving a new series expansion for this function

$$\operatorname{arctanh} x = \frac{y}{x}\left(1 - \frac{2}{3}y + \frac{2.4}{3.5}y^2 - \frac{2.4.6}{3.5.7}y^3 + \cdots\right) \qquad \text{with} \quad y = \frac{x^2}{1-x^2}.$$

Applied to the relation $\log 2 = 2 \operatorname{arctanh} \frac{1}{3}$ and after some easy manipulations comes a new infinite series.

Corollary 13 *The constant* $\log 2$ *may be obtained by the hypergeometric series:*

$$\log 2 = \frac{3}{4}\left(1 - \frac{1}{12} + \frac{1.2}{12.20} - \frac{1.2.3}{12.20.28} + \cdots\right) = \frac{3}{4}\sum_{k\geq 0} \frac{(-1)^k k!^2}{2^k(2k+1)!}. \qquad (30)$$

Each new term of this series is multiplied by $-\frac{k}{8k+4}$ where $k = 1, 2, ...$The same kind of formula was given by Euler to compute the constant π. One nice application of this corollary is to make it feasible to write a tiny code to compute $\log 2$ just like it was done for π. A few tiny codes are available at [15].

7 $\log 2$ and the AGM

The rate of convergence of all the previous series were logarithmic or geometric. Nevertheless, there exist for $\log 2$, just as for π, sequences that have *quadratic* or *quartic* convergence rate and based on the celebrated AGM. (Arithmetic-Geometric Mean [5])

The most classical among the AGM based quadratic convergence algorithms for $\log 2$ is the following. Starting from a_0 and $b_0 > 0$, we consider the iteration

$$\begin{cases} a_{n+1} = \frac{1}{2}(a_n + b_n) \\ b_{n+1} = \sqrt{a_n b_n} \end{cases}$$

and we define the function R by

$$\mathrm{R}(a_0,\,b_0) = \left(\frac{1}{1 - \sum_{n\geq 0} 2^{n-1}(a_n^2 - b_n^2)}\right).$$

Theorem 14 *Let an integer $N \geq 3$ and let $\frac{1}{2} \leq x \leq 1$ a real number, we have*

$$\left|\log x - \mathrm{R}(1,\, 10^{-N}) + \mathrm{R}(1,\, 10^{-N}x)\right| \leq \frac{N}{10^{2(N-2)}}. \tag{31}$$

Proof. See [5] where a quartic similar algorithm is also described. ∎

By choosing $x = \frac{1}{2}$, this algorithm gives about $2N$ decimal digits of $\log 2$. The convergence is quadratic and should be stopped when $2^{n-1}(a_n^2 - b_n^2)$ is less than 10^{-2N} and it will occur for $n \approx \log_2 N$. By mean of fast algorithms based on FFT multiplication to compute square roots, its complexity is $O(n \log^2 n)$ to obtain n digits of $\log 2$.

Notice that this algorithm is not specific to the computation of $\log 2$ and can be used to evaluate $\log x$ for any real number x. It is sufficient to write $x = 2^E m$ where $E \in \mathbf{Z}$ and $\frac{1}{2} < m \leq 1$ and the logarithm is computed thanks to $\log x = \log m + E \log 2$.

Example 15 *Decompositions:*

$$\begin{aligned}
\log 3 &= 2\log 2 + \log\left(\frac{3}{4}\right), \\
\log 5 &= 3\log 2 + \log\left(\frac{5}{8}\right) = 2\log 2 - \log\left(\frac{4}{5}\right), \\
\log 7 &= 3\log 2 + \log\left(\frac{7}{8}\right).
\end{aligned}$$

8 Computation of $\log k$ for successive values

It may be useful to compute to an important accuracy the logarithms of all integers up to a given limit n. This is the case, for example, if we need to evaluate the following infinite series that are related to the derivative of the *Zeta function*:

$$\begin{aligned}
\zeta_a'(s) &= \sum_{k\geq 2} (-1)^{k-1} \frac{\log k}{k^s}, \qquad s > 0, \\
\log 2\left(\gamma - \frac{\log 2}{2}\right) &= \sum_{k\geq 2} (-1)^{k-1} \frac{\log k}{k}.
\end{aligned}$$

The computation of those alternating series can be done to a relatively large accuracy by taking in account only a few terms and therefore by computing just a few logarithms. (See [15] for the description of some fast algorithms to evaluate alternating series.)

8.1 Algorithm with storage of a single logarithm

Starting with a value of $\log 2$, computed by one of the methods described so far, it is easy to find the logarithms of any successive set of integers thanks to the relation

$$2\,\mathrm{L}\,(2k-1) = \log\left(\frac{1+\frac{1}{2k-1}}{1-\frac{1}{2k-1}}\right) = \log\left(\frac{k}{k-1}\right),$$

thus we deduce that:

Theorem 16 *Let $k > 1$ a real number, then*

$$\begin{aligned} \log k &= \log(k-1) + 2\,\mathrm{L}(2k-1) && (32)\\ &= \log(k-1) + 2\left(\frac{1}{2k-1} + \frac{1}{3}\frac{1}{(2k-1)^3} + \frac{1}{5}\frac{1}{(2k-1)^5} + \cdots\right). && (33)\end{aligned}$$

This recurrence may be quite efficient when it is required to compute the logarithms of many successive integers. We observe that the computational effort is decreasing with k and only the storage of $\log(k-1)$ is necessary to compute $\log k$ [19].

Example 17 *With $k = 3$, $k = 5$ and $k = 7$, we obtain respectively*

$$\begin{aligned} \log 3 &= \log 2 + 2\,\mathrm{L}(5)\\ \log 5 &= 2\log 2 + 2\,\mathrm{L}(9),\\ \log 7 &= 3\log 2 + 2\,\mathrm{L}(13).\end{aligned}$$

8.2 Algorithm with storage of two logarithms

We a little more storage, we can improve the previous method thanks to the new relation.

Theorem 18 *Let $k > 2$ a real number, then*

$$\log k = 2\log(k-1) - \log(k-2) - 2\,\mathrm{L}(2k^2 - 4k + 1). \qquad (34)$$

To use this result, the storage of the two logarithms $\log(k-1)$ and $\log(k-2)$ is necessary to compute $\log k$. The amount of work can be reduced if the logarithms of the first small primes (p_1, p_2, ..., p_m) are also stored and using the prime decomposition for all integers k such as $k = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ so that

$$\log k = e_1 \log p_1 + e_2 \log p_2 + \cdots + e_m \log p_m.$$

Example 19 *For $k = 3$, $k = 5$ and $k = 7$, we find*

$$\begin{aligned} \log 3 &= 2\log 2 - 2\,\mathrm{L}(7),\\ \log 5 &= 2\log 4 - \log 3 - 2\,\mathrm{L}(31),\\ \log 7 &= 2\log 6 - \log 5 - 2\,\mathrm{L}(71).\end{aligned}$$

8.3 Algorithm with storage of logarithms of primes

If we are able to store all the logarithms $\log p$ of the sequence of primes we can improve again the efficiency. In his computation [6], Briggs adopted a similar method.

Theorem 20 *Let p a prime (or any non null integer greater than 2), then*

$$\log p = \log 2 + \frac{1}{2}\log\left(\frac{p-1}{2}\right) + \frac{1}{2}\log\left(\frac{p+1}{2}\right) + \mathrm{L}(2p^2-1). \tag{35}$$

Proof. It is a direct consequence of the trivial decomposition

$$p = 2\sqrt{\frac{p-1}{2}}\sqrt{\frac{p+1}{2}}\sqrt{\frac{p^2}{p^2-1}}.$$

■

With the same examples more efficient relations are now deduced.

Example 21 *Let $p = 3$, $p = 5$ and $p = 7$, then*

$$\begin{aligned}
\log 3 &= \frac{3}{2}\log 2 + \mathrm{L}(17), \\
\log 5 &= \frac{3}{2}\log 2 + \frac{1}{2}\log 3 + \mathrm{L}(49), \\
\log 7 &= 2\log 2 + \frac{1}{2}\log 3 + \mathrm{L}(97).
\end{aligned}$$

The calculation of the logarithms of all integers up to $n = 100$ requires to compute $\log 2$ and $\mathrm{L}(k)$ for the following 24 increasing values of k:

17	49	97	241	337	577	721	1057	1681	1921	2737	3361
3697	4417	5617	6961	7441	8977	10081	10657	12481	13777	15841	18817

The computational effort is therefore $E(17, 49, 97, ..., 18817) \approx 4.002$ and it can be compared with the effort $E = 1.048$ required to compute $\log 2$ with the relatively efficient relation $2\,\mathrm{L}(3)$. This can be improved a little by observations, all deduced from the decomposition theorem, such as

$$\begin{aligned}
\mathrm{L}(17) &= 2\,\mathrm{L}(35) + \mathrm{L}(577), \\
\mathrm{L}(49) &= 2\,\mathrm{L}(97) - \mathrm{L}(4801), \\
\mathrm{L}(97) &= 2\,\mathrm{L}(195) + \mathrm{L}(18817),
\end{aligned}$$

and the effort drops to $E = 3.726$.

9 Records of computation

Computing the constant $\log 2$ and other logarithms has inspired a few authors and especially since 1997.

Exact digits	Year	Author	Method
5	~1615	Napier	[25]
7	1624	Kepler	[18]
16	~1671	Newton	Formula (13), [26]
25	1748	Euler	Formula (21), [12]
48	1778	Wolfram	Several logarithms
137	1853	Shanks	Several logarithms, [31]
260	1878	Adams	Several logarithms, [2]
330	1940	Uhler	Several logarithms, [34]
3,683	1962	Sweeney	Formula (14), [33]
2,015,926	1997	Demichel	Formula (21)
5,039,926	1997	Demichel	Formula (21)
10,079,926	1997	Demichel	Formula (21)
29,243,200	12-1997	Gourdon	AGM algorithm (31)
58,484,499	12-1997	Gourdon	AGM algorithm (31)
108,000,000	09-1998	Gourdon	Formulas (26) and (27)
200,001,000	09-2001	Gourdon & Kondo	Formulas (25) and (24)
240,000,000	09-2001	Gourdon & Sebah	Formulas (25) and (29)
500,000,999	09-2001	Gourdon & Kondo	Formulas (25) and (29)
600,001,000	03-2002	Gourdon & Kondo	Formulas (25) and (29)
15,500,000,000	01-2009	Chan & Yee	Formulas (25) and (26)
31,026,000,000	02-2009	Chan & Yee	Formulas (25) and (26)

A Approximations

From the *continued fraction algorithm*, we may extract the two useful rational approximations:

$$\log 2 \approx \frac{7050}{10171} = 0.69314718(316...),$$
$$\log 2 \approx \frac{49180508}{70952475} = 0.693147180559945(230...),$$

with respectively 8 and 15 correct digits. The next approximations, with radicals, are curiosities involving, respectively, only the figures 1 and 3 and the figures 2 and 5:

$$\log 2 \approx \frac{13}{31}\sqrt{1+\sqrt{3}} = 0.69314(811...),$$
$$\log 2 \approx \left(\frac{2}{5}\right)^{\frac{2}{5}} = 0.4^{0.4} = 0.69314(484...).$$

B List of formulas

B.1 Integrals

$$\log 2 = \int_1^2 \frac{dt}{t} = \int_0^1 \frac{dt}{1+t}$$

$$2\,(1-\log 2) = \int_0^1 \frac{dt}{1+\sqrt{t}}$$

$$\log 2 = \frac{9}{16} + 2\int_0^{3/4} \left(\sqrt{1+t^2}-1\right) dt$$

$$\log 2 = \frac{15}{16} - 8\int_0^{1/8} \sqrt{t+t^2}dt$$

$$\log 2 = \int_0^1 \frac{t^{2n+1}-t^n}{\log t}dt, \qquad n \geq 0$$

$$\log 2 - 1 = \int_0^{\pi/2} \sin t \log\left(\sin t\right) dt$$

$$\log 2 = \int_0^\infty \frac{\cos t - \cos(2t)}{t}dt$$

$$\log 2 = \int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt$$

$$\log 2 = \int_0^\infty \frac{dt}{1+e^t}$$

$$2\log 2 = \int_0^\infty \frac{t}{\cosh t + 1}dt$$

$$\frac{\pi}{2}\log 2 = \int_0^1 \frac{\arcsin t}{t}dt$$

$$\frac{\pi}{2}\log 2 = \int_0^\infty \frac{\arctan(2t) - \arctan t}{t}dt$$

$$-\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log\left(\sin t\right) dt$$

$$\frac{\pi}{8}\log 2 = \int_0^{\pi/4} \log(1+\tan t)dt$$

$$2 - 2\log 2 - \frac{\pi^2}{12} = \int_0^1 \log t \log(1+t)dt$$

$$-\frac{\log^2 2}{2} = \int_0^{1/2} \frac{\log(1-t)}{1-t}dt$$

$$\frac{\log^2 2}{2} - \frac{\pi^2}{12} = \int_0^{1/2} \frac{\log(1-t)}{t}dt$$

B.2 Series

B.2.1 Logarithmic series

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \qquad \text{(Mercator-Mengoli)}$$

$$\log 2 = \sum_{k\geq 1} \frac{1}{2k(2k-1)} \qquad \text{(Mercator-Mengoli)}$$

$$\log 2 = -\frac{\sqrt{3}}{3}\pi + 3\left(1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots\right)$$

$$\log 2 = \frac{3}{4} - \frac{1}{4}\sum_{k\geq 1} \frac{1}{k(k+1)(2k+1)}$$

$$\log 2 = \frac{1}{2} + \sum_{k\geq 0} \frac{1}{(2k+1)(2k+2)(2k+3)} \qquad \text{(Knopp [19])}$$

$$\log 2 = 1 - 2\sum_{k\geq 0} \frac{(-1)^k}{(2k+1)(2k+2)(2k+3)} \qquad \text{(Knopp [19])}$$

$$\log 2 = \frac{1}{2} + \frac{1}{2}\sum_{k\geq 1} \frac{1}{k(4k^2-1)} \qquad \text{(Ramanujan [19])}$$

$$\log 2 = \frac{2}{3} + \frac{1}{3}\sum_{k\geq 1} \frac{1}{k(16k^2-1)} \qquad \text{(Ramanujan [19])}$$

$$\log 2 = \frac{3}{4} - \frac{1}{2}\sum_{k\geq 1} \frac{1}{k(4k^2-1)^2} \qquad \text{(Knopp [19])}$$

$$\log 2 = \frac{1}{2} + \sum_{k\geq 1} \frac{(-1)^{k-1}}{k(4k^4+1)} \qquad \text{(Glaisher [19])}$$

$$\log 2 = \frac{1327}{1920} + \frac{45}{4}\sum_{k\geq 4} \frac{(-1)^k}{k(k^2-1)(k^2-4)(k^2-9)} \qquad \text{(Glaisher [14])}$$

$$\log 2 = \frac{5}{6} - 6\sum_{k\geq 0} \frac{(-1)^k}{(2k+1)(2k+2)(2k+4)(2k+5)} \qquad \text{(Knopp [19])}$$

$$\log 2 = \frac{1}{2}\left(\frac{1}{2} + \frac{1.3}{2.4}\frac{1}{2} + \frac{1.3.5}{2.4.6}\frac{1}{3} + \cdots\right) \qquad \text{(Nielsen [27])}$$

$$\log^2 2 = 2\sum_{k\geq 1} \frac{(-1)^{k-1}}{k+1} H_k, \quad H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \qquad \text{([17])}$$

$$\log^2 2 = \sum_{k\geq 0} (-1)^k \left(\frac{1}{(k+1).1} + \frac{1}{k.2} + \cdots + \frac{1}{1.(k+1)}\right) \qquad \text{(Hyslop [16])}$$

$$\frac{\pi}{8}\log 2 = \sum_{k\geq 1} \frac{(-1)^{k-1}}{2k+1} H_{2k} \qquad \text{(Hyslop [16])}$$

$$-4\log 2 = \sum_{\substack{i,j,k,\ell=-\infty \\ (i,j,k,\ell)\neq(0,0,0,0)}}^{\infty} \frac{(-1)^{i+j+k+\ell}}{i^2+j^2+k^2+\ell^2} \qquad ([5])$$

$$\log 2 = \lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$$

$$\log 2 = \alpha \lim_{n\to\infty}\left(\frac{1}{n^\alpha+1^\alpha}+\frac{2^{\alpha-1}}{n^\alpha+2^\alpha}+\cdots+\frac{n^{\alpha-1}}{n^\alpha+n^\alpha}\right), \qquad \alpha>0$$

$$\frac{1}{6}\log 2 = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2k+2n-1}{3\left(2k+2n-1\right)^2-1}\right)$$

$$\log 2 = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n\left(n+1\right)}}+\frac{1}{\sqrt{\left(n+1\right)\left(n+2\right)}}+\cdots+\frac{1}{\sqrt{2n\left(2n+1\right)}}\right)$$

B.2.2 Geometric series

$$\log 2 = \sum_{k\geq 1}\frac{\zeta(2k)-1}{k}$$

$$\log 2 = \frac{1}{2}+\sum_{k\geq 2}\frac{\zeta(k)-1}{2^k}$$

$$\log 2 = \frac{31}{45}+\frac{1}{3}\sum_{k\geq 1}\frac{\zeta(2k+1)-1}{16^k}$$

$$\log 2 = \frac{3}{4}\sum_{k\geq 0}\frac{(-1)^k k!^2}{2^k(2k+1)!}$$

$$\log 2 = \sum_{k\geq 1}\frac{1}{k2^k}$$

$$\log 2 = \frac{1}{2}\sum_{k\geq 1}\frac{H_k}{2^k}, \quad H_k = 1+\frac{1}{2}+\cdots+\frac{1}{k}$$

$$\log^2 2 = \sum_{k\geq 1}\frac{H_k}{k+1}\frac{1}{2^k}$$

$$\log 2 = \frac{2}{3}\sum_{k\geq 0}\frac{1}{(2k+1)9^k}$$

$$\log 2 = \frac{16}{27}\left(1+\left(1+\frac{1}{3}\right)\frac{1}{9}+\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{1}{9^2}+\cdots\right)$$

$$\log 2 = \frac{2}{3}+\sum_{k\geq 1}\left(\frac{1}{2k}+\frac{1}{4k+1}+\frac{1}{8k+4}+\frac{1}{16k+12}\right)\frac{1}{16^k}$$

$$\log 2 = \frac{3}{4} + \frac{1}{4}\sum_{k\geq 1} \frac{(-1)^k(5k+1)(2k)!}{k(2k+1)16^k(k!)^2} \qquad \text{(Knopp [19])}$$

$$\log 2 = 7\log\left(\frac{10}{9}\right) - 2\log\left(\frac{25}{24}\right) + 3\log\left(\frac{81}{80}\right) \qquad \text{(Adams [2])}$$

$$\log 2 = \sqrt{2}\left(\frac{1}{2} - \sum_{k\geq 1} \frac{1}{(4k^2-1)\,(17+12\sqrt{2})^k}\right)$$

$$\log 2 = 2\sum_{k\geq 1} \frac{1}{kP_{k-1}(3)P_k(3)} \qquad \text{(Burnside [9])}$$

In Burnside's formula $P_k(x)$ are *Legendre Polynomials* [1].

B.2.3 Machin like series

The following table is a small selection of fast converging formulas to compute $\log 2$ as linear combinations of $\mathrm{L}(k)$ functions defined by (20). To give an idea of the efficiency the Lehmer's measure E is also computed for each formula. (Most of those relations are extracted from [29].)

$\log 2$	E
$2\,\mathrm{L}(3)$	1.048
$2\,\mathrm{L}(5) + 2\,\mathrm{L}(7)$	1.307
$4\,\mathrm{L}(\frac{29}{5}) - 2\,\mathrm{L}(577)$	0.836
$4\,\mathrm{L}(6) + 2\,\mathrm{L}(99)$	0.893
$4\,\mathrm{L}(7) + 2\,\mathrm{L}(17)$	0.998
$6\,\mathrm{L}(9) + 2\,\mathrm{L}(\frac{253}{3})$	0.784
$10\,\mathrm{L}(17) + 4\,\mathrm{L}(\frac{499}{13})$	0.722
$8\,\mathrm{L}(11) - 4\,\mathrm{L}(111) - 2\,\mathrm{L}(19601)$	0.841
$10\,\mathrm{L}(17) + 8\,\mathrm{L}(79) + 4\,\mathrm{L}(1351)$	0.830
$14\,\mathrm{L}(23) + 6\,\mathrm{L}(65) - 4\,\mathrm{L}(485)$	0.829
$18\,\mathrm{L}(26) - 2\,\mathrm{L}(4801) + 8\,\mathrm{L}(8749)$	0.616
$14\,\mathrm{L}(31) + 10\,\mathrm{L}(49) + 6\,\mathrm{L}(161)$	0.858
$36\,\mathrm{L}(52) - 2\,\mathrm{L}(4801) + 8\,\mathrm{L}(8749) + 18\,\mathrm{L}(70226)$	0.657
$72\,\mathrm{L}(127) + 54\,\mathrm{L}(449) + 34\,\mathrm{L}(4801) - 10\,\mathrm{L}(8749)$	0.689
$144\,\mathrm{L}(251) + 54\,\mathrm{L}(449) - 38\,\mathrm{L}(4801) + 62\,\mathrm{L}(8749)$	0.659
$342\,\mathrm{L}(575) + 198\,\mathrm{L}(3361) + 222\,\mathrm{L}(8749) + 160\,\mathrm{L}(13121) + 106\,\mathrm{L}(56251)$	0.676

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, (1964)

[2] J.C. Adams, *On the value of Euler's constant*, Proc. Roy. Soc. London, (1878), vol. 27, pp. 88-94

[3] D.H. Bailey, P.B. Borwein and S. Plouffe, *On the Rapid Computation of Various Polylogarithmic Constants*, Mathematics of Computation, (1997), vol. 66, pp. 903-913

[4] A. Baker, *A Transcendental Number Theory*, Cambridge University Press, London, (1975)

[5] J.M. Borwein and P.B. Borwein, *Pi and the AGM - A study in Analytic Number Theory and Computational Complexity*, A Wiley-Interscience Publication, New York, (1987)

[6] H. Briggs, *Arithmetica logarithmica sive logarithmorum Chiliades Triginta*, London, (1624)

[7] E.M. Bruins, *On the history of logarithms: Bürgi, Napier, Briggs, de Decker, Vlacq, Huygens*, Janus 67, (1980), vol. 4, pp. 241-260

[8] J. Bürgi, *Arithmetische und geometrische Progress Tabulen, sambt gründlichem unterricht wie solche nützlich in allerley Rechnungen zugerbrauchen und verstanden werden sol*, Prague, (1620)

[9] W. Burnside, *On rational approximations to* $\log x$, Messenger, (1917), vol. 47, pp. 79-80

[10] C.W. Clenshaw and A.R. Curtis, *A method for numerical integration on an automatic computer*, Num. Math., (1960), vol. 2, pp. 197-205

[11] H. Cohen, *A Course in Computational Algebraic Number Theory*, Springer-Verlag, 1995 (Second edition)

[12] L. Euler, *Introduction à l'analyse infinitésimale* (french traduction by Labey), Barrois, aîné, Librairie, (original 1748, traduction 1796), vol. 1, pp. 89-90

[13] G.A. Gibson, *Napier Tercentenary Celebration: Handbook of the Exhibition*, Royal Society of Edinburgh, (1914)

[14] J.W.L. Glaisher, *Methods of increasing the convergence of certain series of reciprocals*, Quart. J., (1902), vol. 34, pp. 252-347

[15] X. Gourdon and P. Sebah, *Numbers, Constants and Computation*, `http://numbers.computation.free.fr/Constants/constants.html`, (1999)

[16] J.M. Hyslop, *Infinite Series*, Dover, New York, (republication 2006, original 1959)

[17] W.J. Kaczor and M.T. Nowak, *Problems in mathematical analysis I: Real numbers, sequences and series*, AMS, (2000)

[18] J. Kepler, *Chilias Logarithmorum*, Marburg, (1624)

[19] K. Knopp, *Theory and application of infinite series*, Blackie & Son, London, (1951)

[20] C. Lanczos, *Applied Analysis*, Dover Publications, New York, (1988, first edition 1956)

[21] D.H. Lehmer, *On Arctangent Relations for* π, The American Mathematical Monthly, (1938), pp. 657-664

[22] E. Maor, *e: The Story of a Number*, Princeton University Press, (1994)

[23] N. Mercator, *Logarithmotechnia: sive methodus construendi logarithmos nova, accurata & facilis*, London, (1668)

[24] J. Napier, *Mirifici logarithmorum canonis descriptio*, Edinburgh, (1614)

[25] J. Napier, *Mirifici logarithmorum canonis constructio*, Edinburgh, (1619)

[26] I. Newton, *Methodus fluxionum et serierum infinitarum*, (1664-1671)

[27] N. Nielsen, *Om log(2) og* $1/1^2 - 1/3^2 + 1/5^2 - 1/7^2 + ...$, Nyt Tidss. for Math., (1894), pp. 22-25

[28] G. Saint-Vincent, *Opus geometricum quadraturae circuli et sectionum coni*, Antwerp, (1647)

[29] P. Sebah, *Machin like formulae for logarithm*, Unpublished, (1997)

[30] L. Seidel, *Ueber eine Darstellung des Kreisbogens, des Logarithmus und des elliptischen Integrales erster Art durch unendliche Producte*, Borchardt J., (1871), vol. 73, pp. 273-291

[31] W. Shanks, *Contributions to Mathematics Comprising Chiefly the Rectification of the Circle to 607 Places of Decimals*, G. Bell, London, (1853)

[32] W. Shanks, *Second paper on the numerical values of* $e, \log e^2, \log e^3$ *and* $\log e^{10}$, *also on the numerical value of* M *the modulus of the common system of logarithms, all to 205 decimals*, Proc. of London, (1871), vol. 19, pp. 27-29

[33] D.W. Sweeney, *On the Computation of Euler's Constant*, Mathematics of Computation, (1963), pp. 170-178

[34] H.S. Uhler, *Recalculation and extension of the modulus and of the logarithms of 2, 3, 5, 7 and 17*, Proc. Nat. Acad. Sci., (1940), vol. 26, pp. 205-212

[35] A. Vlacq, *Arithmetica logarithmica*, Gouda, (1628)

[36] K. Weierstrass, *Zu Lindemann's Abhandlung: 'Über die Ludolph'sche Zahl'*, Sitzungber. Königl. Preuss. Akad. Wissensch. zu Berlin, (1885), vol. 2, pp. 1067-1086