Classification of numbers : overview

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Abstract

Elementary facts on the classification of real numbers. A usual classification of numbers is to divide them in different categories and, for example, we have the following relations:

$$\text{Integers} \subset \text{Rational Numbers} \subset \text{Algebraic Numbers}$$

1 Rational and irrational numbers

Definition 1 *Rational numbers can be expressed as fraction of integers $r = p/q$, where (p, q) are integers (p is the numerator and q the denominator).*

Definition 2 *The set of rational numbers is denoted* $\mathbf{Q}$.

Numbers like $7/5, 1/1999, 31416/10000$ are rational numbers. During ancient time the mathematicians believed that all numbers were rational, Pythagoreans (a mathematical and philosophical school in Greece during the VI-th century B.C.) based their system on such a belief. It was a scandal when they proved that $\sqrt{2}$ could not be written as a fraction and it's why such numbers were called *irrational*.

Rational numbers can be defined by a finite amount of information (the numerator and denominator), their complexity is finite.

A basic property of rational numbers is that their decimal expansion is finite (regular numbers) or periodic (non regular numbers). Let us consider:

$$\begin{aligned} \frac{87}{32} &= 2.71875 \\ \frac{228}{395} &= 0.5\underline{7721518987341}77215189873417721518987341... \end{aligned}$$

we see that the first example is a regular rational number and the second is a non regular one with a period $L = 7721518987341$ and the length of this period is 13.

We have the property, if λ denotes the length of the period and if p and q are relatively prime, that $\lambda(p/q) = \lambda(1/q)$ for any non regular number and that $\lambda(1/q)$ is at most $q-1$. And the previous case becomes:

$$\lambda\left(\frac{228}{395}\right) = \lambda\left(\frac{1}{395}\right) = 13.$$

After $\sqrt{2}$ it was a new challenge for mathematicians to find other irrational numbers among the famous constants. Let us quote some major results:

- e is irrational (Euler 1744)
- π is irrational (Lambert 1761 [9] and Legendre 1794 [10])
- $\zeta(3)$ is irrational (Apery 1978 [13])
- $\zeta(2k+1)$ are irrational for infinitely many integers k (Rivoal 2000 [14])

Some irrationality proofs of classical constants can be found in [5] or [6].

2 Algebraic numbers

Irrational numbers can be classified in two categories, the *Algebraic numbers* and the *Transcendental numbers*.

Definition 3 *An algebraic number x is the root of a polynomial equation with integers coefficients :*

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0.$$

If n is the smallest possible degree of the polynomial, x is called an algebraic number of *degree n*. For example, the algebraic numbers of degree $n = 1$ define the set of rational numbers. Square roots of integers which are not perfect squares are algebraic numbers of degree 2. All the solutions of the following equations are algebraic numbers of degree 2 or 3 for the last example:

$$\begin{aligned} x^2 - 2 &= 0 \\ x^2 - x - 1 &= 0 \\ x^2 + 8x - 35 &= 0 \\ 90x^3 - 3x^2 - 4x - 14 &= 0 \end{aligned}$$

In other words, $\sqrt{2}, \varphi = (1+\sqrt{5})/2, \sqrt{51} - 4, \ldots$ are algebraic numbers.

In a certain sense, algebraic numbers are represented by a finite collection of integers (the coefficients $(a_0, a_1, \ldots, a_n)$ of the polynomial), so their complexity may be considered as finite.

It's possible to show that the sum, the product, the quotient of algebraic numbers are also algebraic numbers (in mathematical terms, the set of algebraic numbers which is sometimes denoted $\mathbf{A}$, is a field).

3 Transcendental numbers

3.1 Elementary results

Definition 4 *A numbers which is not root of any polynomial equation with integer coefficients is called a transcendental number.*

Definition 5 *Transcendental numbers are non algebraic numbers.*

In 1844, Joseph Liouville (1809-1882) [12] was the first to prove the existence of such numbers and he gave an important and elementary characterization of a category of transcendental numbers, the *Liouville numbers*:

Theorem 6 *The non rational number x is a Liouville number if, for every positive integer n there exist two integers (p, q) such as*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^n} \qquad q > 1,$$

and such a number is transcendental.

He used this result to exhibit the first concrete transcendental numbers. One of those numbers is famous

$$L = \sum_{k=1}^{\infty} \frac{1}{10^{k!}} = 0.110001000000000000000001000 0...,$$

it has 0 everywhere except a few 1 at position $k!$ $(1, 2, 6, 24, 120, 720, ...)$. This number is not rational because it's clearly not periodic and satisfies the characterization theorem because there so many 0 between the 1 that it can be approximated by a rational number very easily. All transcendental numbers are not Liouville numbers, in 1953 Mahler proved that the constant π is not a Liouville number. He also proved that non rational algebraic numbers are not easy to approximate with rational numbers.

Theorem 7 *For any algebraic and non rational number x of degree n, there exist a positive constant C, such as for any couple of integers (p, q) with $q > 0$*

$$\left|x - \frac{p}{q}\right| > \frac{1}{Cq^n}.$$

In 1873, Georg Cantor (1845-1918) proved that the transcendental numbers are non denumerable, whereas the algebraic numbers are denumerable. In other words, there are *much more* transcendental numbers than algebraic numbers. Because the proof is elementary we give it.

Theorem 8 *The set of algebraic numbers is denumerable.*

Proof. For any polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ with integer coefficients of degree n, define it's height $h(P)$ as the integer

$$n - 1 + |a_n| + |a_{n-1}| + ... + |a_0|$$

and let T_m be the set of all the roots of all the polynomials P such as $h(P) \leq m$. It's clear that for any integer m the number of polynomials with $h(P) \leq m$ is finite, hence the set T_m is finite (the number of roots of a polynomial is finite). Finally the set of algebraic numbers is

$$\bigcup_{m=1}^{\infty} T_m$$

and because a denumerable union of finite sets is denumerable the theorem follows. ■

Proving that a given number is transcendental is usually a very hard problem. The two most famous historical results are :

- e is transcendental (Hermite 1873 [7])
- π is transcendental (Lindemann 1882 [11], Weierstrass 1885, Hilbert 1893 [8])

An important property of those numbers is the impossibility to construct any transcendental number with geometric construction (with only the help of a compass and a ruler). The celebrated problem of *squaring the circle* which goes back to ancient times and occupied generations of mathematicians was showed to have no solution with the proof of the transcendence of π.

3.2 Modern results

In 1934, Aleksandr Gelfond (1906-1968) and Theodor Schneider (1911-) independently solved one of the famous Hilbert's problems (*the Seventh Problem: "Irrationality and transcendence of certain numbers"*) which were cited at the International Congress of Mathematicians at Paris in 1900.

Theorem 9 *(Gelfond-Schneider). Let α an algebraic number ($\alpha \neq 0, \alpha \neq 1$) and β an algebraic and irrational number, then*

$$\alpha^{\beta} \text{ is transcendental.}$$

Corollary 10 *The numbers $2^{\sqrt{2}}$(the Gelfond-Schneider constant) and e^{π} are transcendental numbers.*

In 1966, Alan Baker (1939-) succeeded in giving a vast generalization of Gelfond-Schneider theorem : any finite sum (if not zero)

$$\sum_i a_i \log(b_i)$$

is transcendental when (a_i, b_i) are algebraic numbers with a_i non zero and b_i different from 1. As a consequence the number $\log(2)$ is transcendental, as for the logarithm of any integer or rational number different from 1.

The sum of a transcendental number and an algebraic number is also transcendental [2]. Some isolated results are available:

- $\Gamma(1/3)$ is transcendental (Le Lionnais 1983) where Γ is the *gamma function*.
- $\Gamma(1/4)$ and $\Gamma(1/6)$ are transcendental (Chudnovsky 1984)
- $\log(3)/\log(2)$ is transcendental (Hardy & Wright 1979 [6])
- Champernowne's number [4] defined by writing the consecutive integers 0.12345678910111213... was proven to be transcendental by Mahler in 1961.

4 Non yet classified numbers

It is not known whether $\pi + e$, γ (Euler's constant), B (Brun's constant), $\zeta(3)$, ... are transcendental. We don't even know if $\gamma, \zeta(5)$ nor G (Catalan's constant) are irrational.

Only numerical estimation are available, Brent & McMillan showed in 1980 [3] that if $\gamma = p/q$ then $q > 10^{15000}$. This result has been improved to $q > 10^{242080}$ by Papanikolaou in 1997.

In 1988, Bailey also established that no polynomial with small integer coefficients (of Euclidian norm smaller than 10^9) of degree 8 or less exists which vanishes on the constants $\pi + e$, $\gamma, e^\gamma, ...$ [1].

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