Convergence acceleration of series

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Abstract

We propose an overview of some celebrated methods to accelerate the convergence of numerical series. It includes some examples to appreciate the abilities and limits of each algorithm.

The sum of an infinite series whose final term vanishes perhaps is infinite, perhaps finite (in *Ars Conjectandi*).

- Jakob Bernoulli (1654-1705).

With the exception of the geometric series there does not exist in all of mathematics a single infinite series whose sum has been determined rigorously (quoted by Eli Maor [11]).

- Niels Henrik Abel (1802-1829).

1 Introduction

Numerous mathematical constants are calculated as limit of series and many of those series are very slow to converge requiring therefore methods to accelerate their convergence. The reader may consult [4] for more justifications of some of the algorithms described here and [9] for a good mathematical background on series and the concept of convergence.

In this section, we recall some definitions and elementary results on well known series. For any series $\sum_k a_k$, we denote by s_n its partial sums, that is

$$\left| \begin{array}{l} s_0 = a_0, \\ s_1 = a_0 + a_1, \\ s_n = a_0 + a_1 + \ldots + a_n = \sum_{k=0}^{n} a_k. \end{array} \right.$$

and we are interested in the behavior of s_n as n tends to infinite (that is *infinite series*). The series whose terms have alternative signs are said to be *alternating series*, we write them as

$$s_n = a_0 - a_1 + \ldots + (-1)^n a_n = \sum_{k=0}^{n} (-1)^k a_k,$$

where the (a_k) are positive numbers.

Definition 1 *An infinite series $\sum_{k=0}^{n} a_k$ is said to be convergent if its partial sums (s_n) tends to a limit S (called the sum of the series) as n tends to infinite. Otherwise the series is said to be divergent.*

The following examples are elementary and occur frequently and it's usual to compare a given series to one of those.

Example 2 *(Geometric series) For the geometric series which is defined by $a_k = x^k$, the partial sum s_n is then given by*

$$s_n = 1 + x + x^2 + ... + x^n = \frac{1 - x^{n+1}}{1 - x},$$

hence it converges for $|x| < 1$ to $S = 1/(1 - x)$ and diverges otherwise.

Example 3 *(Harmonic series) The harmonic series is defined for $k > 0$ by $a_k = 1/k$, and the partial sum s_n (sometimes denoted H_n and called Harmonic number) is*

$$s_n = H_n = 1 + \frac{1}{2} + \frac{1}{3} ... + \frac{1}{n}.$$

We have for $n = 2^p$

$$\begin{aligned} s_n &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + .. + \frac{1}{8}\right) + ... + \left(\frac{1}{2^{p-1}+1} + ... + \frac{1}{2^p}\right) \\ s_n &> 1.\frac{1}{2} + 2.\frac{1}{4} + 4.\frac{1}{8} + ... + 2^{p-1}.\frac{1}{2^p} = \frac{p}{2}, \end{aligned}$$

therefore the partial sums are unbounded and the harmonic series is divergent (this important result was known to Jakob Bernoulli (1654-1705) and a long time before to the french bishop Nicolas Oresme (1323-1382)).

Example 4 *(Riemann Zeta series) A natural generalization of the harmonic series is to consider the partial sum*

$$s_n = 1 + \frac{1}{2^s} + \frac{1}{3^s} + ... + \frac{1}{n^s}$$

which converges to the Riemann Zeta function $\zeta(s)$ for any complex numbers s such as $\Re(s) > 1$. The case $s = 1$ is the Harmonic series and the limits for the even values of s are well known since Euler who established for example that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90, ...$

2 Kummer's acceleration method

Ernst Kummer's (1810-1893) method is very natural to understand and may be used to accelerate the convergence of many series. It goes back to 1837 and the idea is to subtract from a given convergent series $\sum a_k$ another *equivalent* series $\sum b_k$ whose sum $C = \sum_{k\geq 0} b_k$ is well known. More precisely suppose that

$$\lim_{k\to\infty} \left(\frac{a_k}{b_k}\right) = \rho \neq 0,$$

then the transformation is given by

$$\sum_{k=0}^{\infty} a_k = \rho \sum_{k=0}^{\infty} b_k + \sum_{k=0}^{\infty} \left(1 - \rho\frac{b_k}{a_k}\right) a_k = \rho C + \sum_{k=0}^{\infty} \left(1 - \rho\frac{b_k}{a_k}\right) a_k. \qquad (1)$$

The convergence of the right hand side of equation (1) is faster because $1 - \rho b_k/a_k$ tends to 0 as k tends to infinity [1].

Example 5 *Consider*

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \frac{1}{k^2}, \\ C &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1, \end{aligned}$$

we have $\rho = C = 1$ and Kummer's transformation (1) becomes

$$S = 1 + \sum_{k=1}^{\infty} \left(1 - \frac{k^2}{k(k+1)}\right) \frac{1}{k^2} = 1 + \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)}.$$

The process can be repeated with this time

$$\begin{aligned} S^* &= \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} \\ C &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} = \frac{1}{4}, \end{aligned}$$

giving

$$S = \frac{5}{4} + 2\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)}.$$

After N applications of the transformation

$$S = \sum_{k=1}^{N} \frac{1}{k^2} + N! \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)...(k+N)},$$

whose convergence may be rapid enough for suitable values of N (this example is due to Knopp [9]). Note, that we have used the following family of series to *improve the convergence*

$$\begin{aligned} C &= \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k(k+1)...(k+N)} = \frac{1}{N.N!}, \\ b_k &\sim \frac{1}{k^{N+1}} \text{ when } k \to \infty. \end{aligned}$$

3 Richardson Extrapolation method

Suppose that the partial sum verifies the relation

$$S = s_n + \frac{\alpha}{n^p} + O\left(\frac{1}{n^{p+1}}\right), \tag{2}$$

where S is the limit of the sum, p a known integer and α a constant which we don't need to determine. Then, following the ideas of Lewis Fry Richardson (1881-1953) [12], it's natural to define the transformation

$$s_n^{(1)} = \frac{2^p s_{2n} - s_n}{2^p - 1} = s_{2n} + \frac{1}{2^p - 1}\left(s_{2n} - s_n\right) \tag{3}$$

in order to eliminate the term in α from relation (2). The process (3) may be repeated on the new series $s_n^{(1)}$

$$\begin{aligned} s_n^{(2)} &= \frac{2^{p+1} s_{2n}^{(1)} - s_n^{(1)}}{2^{p+1} - 1}, \\ s_n^{(3)} &= \frac{2^{p+2} s_{2n}^{(2)} - s_n^{(2)}}{2^{p+2} - 1}, \\ &\quad ... \end{aligned}$$

This method is also well suited to accelerate the computation of numerical integrals by mean of trapezoidal rule as illustrated by Werner Romberg in [13].

Example 6 *(from [14]) Suppose that you have computed, by geometric means and like Archimedes did, the perimeters of the circumscribed polygon with respectively* $6, 12, 24, 48$ *and* 96 *sides on a unit circle then in this case* $S = \pi, p = 2$ *and Richardson's process becomes here*

$$\begin{aligned} s_n^{(1)} &= s_{2n} + \frac{1}{3}\left(s_{2n} - s_n\right), \\ s_n^{(2)} &= s_{2n}^{(1)} + \frac{1}{15}\left(s_{2n}^{(1)} - s_n^{(1)}\right), \\ s_n^{(3)} &= s_{2n}^{(2)} + \frac{1}{63}\left(s_{2n}^{(2)} - s_n^{(2)}\right), \\ &\quad ... \end{aligned}$$

It produces the approximations (the number of sides is $6n$)

n	s_n	$s_n^{(1)}$	$s_n^{(2)}$	$s_n^{(3)}$
1	3.4641016151...	3.1324865405...	3.1416562605...	3.1415925429...
2	3.2153903091...	3.1410831530...	3.1415935385...	3.1415926532...
4	3.1596599420...	3.1415616394...	3.1415926670...	
8	3.1460862151...	3.1415907278...		
16	3.1427145996...			

and finally we find $\pi \approx s_1^{(4)} = 3.141592653(637...)$ *which is correct to nearly 10 digits. With a little supplement of efforts and with the same datas he computed, it is therefore easy to improve the 2 digits value estimated by Archimedes !*

4 Aitken's acceleration and related methods

4.1 Aitken's δ^2-process

In 1926, Alexander Aitken (1895-1967) found a new procedure to accelerate the rate of convergence of a series, the *Aitken δ^2-process* [2]. It consists in constructing a new series $S^{(1)}$, whose partial sums $s_n^{(1)}$ are given by

$$s_n^{(1)} = s_{n+1} - \frac{(s_{n+1} - s_n)^2}{s_{n+1} - 2s_n + s_{n-1}} = \frac{s_{n-1}s_{n+1} - s_n^2}{s_{n+1} - 2s_n + s_{n-1}}, \tag{4}$$

where (s_{n-1}, s_n, s_{n+1}) are three successive partial sums of the original series. Of course, it's possible to repeat the process on the new series $S^{(1)}$ to obtain $S^{(2)}$... This process was in fact known to the Japanese mathematician Takakazu Seki Kowa (1642-1708) who used it to compute 10 correct digits of the constant π by accelerating the convergence of the *polygon algorithm.*

For example for the converging geometric series with $0 < x < 1$:

$$\begin{aligned} s_n &= 1 + x + x^2 + ... + x^n = \frac{1 - x^{n+1}}{1 - x} \text{ and the limit is} \\ S &= \lim_{n\to\infty} s_n = \frac{1}{1-x}, \end{aligned}$$

hence

$$s_n = (1 - x^{n+1})S = S - x^{n+1}S,$$

and if we replace (s_{n-1}, s_n, s_{n+1}) by this expression in (4) we have $s_n^{(1)} = S$ ..., the exact limit is given just by three evaluations of the original series. This indicates that for a series almost geometric, Aitken's process may be very efficient.

Example 7 *If we consider the extremely slow (logarithmic rate) converging series*

$$S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \right) = \log(2),$$

then the following array illustrates the result of three consecutive applications of Aitken's process to the 7 first computed terms of this series :

n	s_n	$s_n^{(1)}$	$s_n^{(2)}$	$s_n^{(3)}$
0	1.0000000000...			
1	0.5000000000...	0.7000000000...		
2	0.8333333333...	0.6904761904...	0.6932773109...	
3	0.5833333333...	0.6944444444...	0.6931057563...	0.6931488693...
4	0.7833333333...	0.6924242424...	0.6931633407...	
5	0.6166666666...	0.6935897435...		
6	0.7595238095...			

A formal expression may be obtained in this case ; the expressions of (s_{n-1}, s_n, s_{n+1}) *are*

$$s_n = s_{n-1} + \frac{(-1)^n}{n+1}, s_{n+1} = s_n + \frac{(-1)^{n+1}}{n+2}$$

so that

$$s_n^{(1)} = s_{n+1} + \frac{(-1)^n (n+1)}{(n+2)(2n+3)}$$

For $n = 1000,$ *it becomes*

$$\begin{aligned} s_{1000} &= 0.69(264...), \\ s_{1000}^{(1)} &= 0.693147180(435...). \end{aligned}$$

4.2 The ε-algorithm

In 1956, Peter Wynn obtained a generalization of Aitken's algorithm which is called the ε-algorithm [15]. It is defined by the transformation

$$\varepsilon_n^{(k+1)} = \varepsilon_n^{(k-1)} + \frac{1}{\varepsilon_{n+1}^{(k)} - \varepsilon_n^{(k)}} \tag{5}$$

starting with $\varepsilon_n^{(-1)} = 0$ and $\varepsilon_n^{(0)} = s_n$ where s_n is the series to be accelerated.

Example 8 *Again we take the series*

$$\lim_{n \to \infty} \left(\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \right) = \log(2),$$

the following array illustrates the result of consecutive applications of Wynn's process (5) to the same as in the previous example 7 first computed terms of the series :

n	$\varepsilon_n^{(-1)}$	$\varepsilon_n^{(0)} = s_n$	$\varepsilon_n^{(1)}$	$\varepsilon_n^{(2)}$	$\varepsilon_n^{(3)}$	$\varepsilon_n^{(4)}$	$\varepsilon_n^{(5)}$
0	0	1.0000000000...	-2	0.7000000000...	-102	0.6933333333...	-3852
1	0	0.5000000000...	3	0.6904761904...	248	0.6930894308...	12015
2	0	0.8333333333...	-4	0.6944444444...	-490	0.6931693989...	
3	0	0.5833333333...	5	0.6924242424...	852		
4	0	0.7833333333...	-6	0.6935897435...			
5	0	0.6166666666...	7				
6	0	0.7595238095...					

and finally $\varepsilon_n^{(6)} = 0.6931524547...$ *Note that the odd values of the index are intermediate quantities.*

5 Euler-Maclaurin summation formula

Suppose that the (a_k) may be written as the image of a given differentiable function f, that is

$$a_k = f(k)$$

and so

$$s_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} f(k)$$

the following famous result due to Leonhard Euler (1732) and Colin Maclaurin (1742, [10]) states that

$$s_n = \int_0^n f(t)dt + \frac{1}{2}\left(f(0) + f(n)\right) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left(f^{2k-1}(n) - f^{2k-1}(0)\right) + \varepsilon_{m,n} \quad (6)$$

where $\varepsilon_{m,n}$ is a remainder given by

$$\varepsilon_{m,n} = \frac{1}{(2m+1)!}\int_0^n B_{2m+1}(x - \lfloor x \rfloor) f^{2m+1}(x)dx$$

and where the B_{2k} are Bernoulli's Numbers and $B_i(x)$ being Bernoulli's polynomials (see Bernoulli's number essay for more details or [8] for a proof).

Example 9 *Suppose that* $s > 1$ *and*

$$f(x) = \frac{1}{(x+1)^s}$$

then (6) becomes

$$\begin{aligned} s_{n-1} &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + ... + \frac{1}{n^s} \\ &= \frac{1}{s-1}\left(1 - \frac{1}{n^{s-1}}\right) + \frac{1}{2}\left(1 + \frac{1}{n^s}\right) + \frac{B_2}{2}\binom{s}{1}\left(1 - \frac{1}{n^{s+1}}\right) + ... \\ &\quad + \frac{B_{2m}}{2m}\binom{s+2m-2}{2m-1}\left(1 - \frac{1}{n^{s+2m-1}}\right) + \varepsilon_{m,n} \end{aligned}$$

if $n \to \infty$ in this identity, we find the relation

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{s-1} + \frac{1}{2} + \frac{B_2}{2}\binom{s}{1} + ... + \frac{B_{2m}}{2m}\binom{s+2m-2}{2m-1} + \varepsilon_{m,\infty}$$

and by computing $\zeta(s) - s_{n-1}$ we find the useful algorithm to estimate $\zeta(s)$

$$\zeta(s) = 1+\frac{1}{2^s}+...+\frac{1}{n^s}+\frac{1}{s-1}\frac{1}{n^{s-1}}-\frac{1}{2n^s}+\sum_{i=1}^{m}\frac{B_{2i}}{2i}\binom{s+2i-2}{2i-1}\frac{1}{n^{s+2i-1}}+(\varepsilon_{m,\infty}-\varepsilon_{m,n}),$$

and for any given integer m, the remainder tends to zero (Exercise : check this with care!).

6 Convergence improvement of alternating series

Very efficient methods exist to accelerate the convergence of an alternating series

$$S = \sum_{k=0}^{\infty} (-1)^k a_k,$$

one of the first is due to Euler. Usually the transformed series converges geometrically with a rate of convergence depending on the method.

Because the speed of converge may be dramatically improved, there is a trick due to Van Wijngaarden which permits to transform any series $\sum b_k$ with positive terms b_k into an alternating series. It may be written

$$\sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} (-1)^k a_k$$

with

$$a_k = \sum_{j=0}^{\infty} 2^j\, b_{2^j(k+1)-1} = b_k + 2b_{2k+1} + 4b_{4k+3} + 8b_{8k+7} + ... \tag{7}$$

and because $2^j(k+1)-1$ increases very quickly with j, only a few terms of the sum (7) are required.

Sometimes a closed form exists for the a_k, for example if we take

$$b_k = \frac{1}{(k+1)^2},$$

then

$$\begin{aligned} a_k &= \sum_{j=0}^{\infty} 2^j b_{2^j(k+1)-1} = \sum_{j=0}^{\infty} \frac{2^j}{\left(2^j(k+1)\right)^2} = \frac{1}{(k+1)^2}\sum_{j=0}^{\infty}\frac{1}{2^j} \\ &= \frac{2}{(k+1)^2}, \end{aligned}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2}.$$

6.1 Euler's method

In 1755, Euler gave a first version of what is now called *Euler's transformation*. To establish it, observe that the sum of the alternating series may be written as

$$2S = a_0 + (a_0 - a_1) - (a_1 - a_2) + (a_2 - a_3) - ... = a_0 + S_1,$$

with

$$S_1 = \Delta^1 a_0 - \Delta^1 a_1 + \Delta^1 a_2 - ... + (-1)^k \Delta^1 a_k + ...,$$

and the *first difference* operator is denoted by

$$\Delta^1 a_k = a_k - a_{k+1}.$$

The new series S_1 is also alternating and the process may be applied again

$$2S_1 = \Delta^1 a_0 + (\Delta^1 a_0 - \Delta^1 a_1) - (\Delta^1 a_1 - \Delta^1 a_2) + (\Delta^1 a_2 - \Delta^1 a_3) - ... = \Delta^1 a_0 + T_1,$$

with this time

$$T_1 = \Delta^2 a_0 - \Delta^2 a_1 + \Delta^2 a_2 - ... + (-1)^k \Delta^2 a_k + ...,$$

and the *second difference* operator is denoted by

$$\Delta^2 a_k = \Delta^1 a_k - \Delta^1 a_{k+1},$$

the following theorem is then easily deduced by repeating the process (we omit some details given in [9]).

Theorem 10 *(Euler) Let $S = \sum_{k=0}^{n} (-1)^k a_k$ be a convergent alternating series, then*

$$S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{2} \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \Delta^k a_0 \right), \tag{8}$$

and Δ is the forward difference operator defined by

$$\Delta^k a_0 = \Delta^{k-1} a_0 - \Delta^{k-1} a_1 = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_j.$$

The first values of $\Delta^k a_0$ are

$$\begin{aligned}
\Delta^0 a_0 &= a_0, \\
\Delta^1 a_0 &= a_0 - a_1, \\
\Delta^2 a_0 &= \Delta^1 a_0 - \Delta^1 a_1 = a_0 - 2a_1 + a_2, \\
& \quad ..., \\
\Delta^k a_0 &= a_0 - k a_1 + \frac{k(k-1)}{2!} a_2 - \frac{k(k-1)(k-2)}{3!} a_3 + ... + (-1)^k a_k.
\end{aligned}$$

Example 11 *Sometimes it's possible to find a closed form for the transformation (8), take*

$$\begin{aligned} S &= \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \sum_{k=0}^{\infty} (-1)^k a_k \text{ with} \\ a_k &= \frac{1}{2k+1}, \end{aligned}$$

the rate of convergence of this famous series is extremely slow, but Euler's method gives

$$\begin{aligned} \Delta^0 a_0 &= 1, \\ \Delta^1 a_0 &= 1 - \frac{1}{3} = \frac{1.2}{1.3}, \\ \Delta^2 a_0 &= \Delta^1 a_0 - \Delta^1 a_1 = \frac{1.2}{1.3} - \frac{1.2}{3.5} = \frac{1.2.4}{1.3.5}, \\ \Delta^3 a_0 &= \Delta^2 a_0 - \Delta^2 a_1 = \frac{1.2.4}{1.3.5} - \frac{1.2.4}{3.5.7} = \frac{1.2.4.6}{1.3.5.7}, \\ & ... \end{aligned}$$

this suggest the expression (prove it by induction ...)

$$\Delta^n a_0 = \frac{1.2.4...(2n)}{1.3.5...(2n+1)}.$$

and

$$\begin{aligned} \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{1}{2}\left(1 + \frac{1.2}{1.3}\frac{1}{2} + \frac{1.2.4}{1.3.5}\frac{1}{2^2} + ...\right) \\ \frac{\pi}{4} &= \frac{1}{2}\left(1 + \frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7}...\right) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!}, \end{aligned}$$

relation also given by Euler.

6.2 Improvement

In 1991, a generalization of Euler's method was given in [5] (this work in fact generalizes a technique that was used to obtain irrationality measures of some classical constants like log(2) for example). This algorithm is very general, powerful and easy to understand. For an alternating series $\sum(-1)^k a_k$, we assume that there exist a *positive* function $w(x)$ such that

$$a_k = \int_0^1 x^k w(x) dx.$$

This relation permits to write the sum of the series as

$$\sum_{k=0}^{\infty} (-1)^k a_k = \int_0^1 \left(\sum_{k=0}^{\infty} (-1)^k x^k w(x) dx \right) = \int_0^1 \frac{w(x)}{1+x} dx.$$

Now, for any *sequence* of polynomials $P_n(x)$ of degree n with $P_n(-1) \neq 0$, we denote by S_n the number

$$S_n = \frac{1}{P_n(-1)} \int_0^1 \frac{P_n(-1) - P_n(x)}{1+x} w(x)dx.$$

Notice that S_n is a linear combination of the number $(a_k)_{0 \leq k < n}$ since if we write $P_n(x) = \sum_{k=0}^n p_k(-x)^k$, we have easily

$$S_n = \frac{1}{P_n(-1)} \sum_{k=0}^{n-1} c_k(-1)^k a_k \qquad \text{with} \qquad c_k = \sum_{j=k+1}^{n} p_j \tag{9}$$

The number S_n satisfies

$$S_n = \int_0^1 \frac{w(x)}{1+x}dx - \int_0^1 \frac{P_n(x)w(x)}{P_n(-1)(1+x)}dx = S - \int_0^1 \frac{P_n(x)w(x)}{P_n(-1)(1+x)}dx$$

and therefore we deduce

$$|S_n - S| \leq \frac{1}{|P_n(-1)|} \int_0^1 \frac{|P_n(x)|\, w(x)}{(1+x)}dx \leq \frac{M_n}{|P_n(-1)|} |S|, \qquad M_n = \sup_{x \in [0,1]} |p_n(x)|.$$

This inequality suggests to choose polynomials with small value of $M_n/|P_n(-1)|$.

6.2.1 Choice of a family of polynomials

1. A first possible choice is

   $$P_n(x) = (1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-x)^k, \qquad \text{for which} \quad P_n(-1) = 2^n, M_n = 1.$$

   It leads to the acceleration

   $$|S_n - S| \leq \frac{|S|}{2^n}$$

   where

   $$S_n = \frac{1}{2^n} \sum_{k=0}^{n-1} (-1)^k c_k a_k, \qquad c_k = \sum_{j=k+1}^{n} \binom{n}{j}.$$

   This choice corresponds in fact to Euler's method.

2. Another choice is

   $$P_n(x) = (1-2x)^n = \sum_{k=0}^{n} 2^k \binom{n}{k} (-x)^k, \quad \text{for which} \quad P_n(-1) = 3^n, M_n = 1.$$

It leads to the acceleration

$$|S_n - S| \leq \frac{|S|}{3^n}$$

where

$$S_n = \frac{1}{3^n}\sum_{k=0}^{n-1}(-1)^k c_k a_k, \qquad c_k = \sum_{j=k+1}^{n} 2^j \binom{n}{j}.$$

3. Chebyshev's polynomials (see [1]) shifted to $[0,1]$ provide a more efficient acceleration. They satisfy the relation

$$P_n(\sin^2 t) = \cos(2nt) \tag{10}$$

and explicitly writes in the form

$$P_n(x) = \sum_{j=0}^{n} \frac{n}{n+j}\binom{n+j}{2j} 4^j (-x)^j.$$

The relation (10) shows that $M_n = 1$ and $|P_n(-1)| \geq \frac{1}{2}(3+\sqrt{8})^n > 5.828^n/2$. This family leads to the following acceleration process :

$$|S_n - S| \leq \frac{2\,|S|}{5.828^n}$$

where

$$S_n = \frac{1}{P_n(-1)}\sum_{k=0}^{n-1}(-1)^k c_k a_k, \qquad c_k = \sum_{j=k+1}^{n} \frac{n}{n+j}\binom{n+j}{2j} 4^j.$$

4. Other families of orthogonal polynomials such as Legendre's polynomials, Niven's polynomial may give interesting accelerations. More details and results may be found in the very interesting paper [5].

6.2.2 Examples

Once the choice of a sequence of polynomials is made, it can be applied to compute the value of many alternating series such as

$$\begin{aligned}
\log(2) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} & a_k &= \frac{1}{k+1} = \int_0^1 x^k\,dx \\
\frac{\pi}{4} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} & a_k &= \frac{1}{2k+1} = \int_0^1 x^k\,\frac{x^{-1/2}}{2}\,dx \\
(1-2^{-s})\zeta(s) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^s}, & a_k &= \frac{1}{(k+1)^s} = \frac{1}{\Gamma(s)}\int_0^1 x^k\,|\log(x)|^{s-1}\,dx.
\end{aligned}$$

Notice that this latest method is very efficient and may be used to compute the value of the zeta function at values of s with $\Re(s) > 0$ (see [3]). Another beautiful alternating series whose convergence can be accelerated in this way is

$$\log(2)\left(\gamma - \frac{1}{2}\log(2)\right) = \frac{\log(2)}{2} - \frac{\log(3)}{3} + \frac{\log(4)}{4} - \frac{\log(5)}{5} + \cdots$$

where γ is the Euler constant (see [7] for a proof of this formula).

7 Acceleration with the Zeta function

If you know how to evaluate the Zeta function at integers values, there is an easy and convenient way to transform your original series in a geometric fast converging one (based on [6]). Suppose that you want to estimate a series which has the form

$$S = A + \sum_{k=2}^{\infty} f\left(\frac{1}{k}\right) \tag{11}$$

where A is a constant and where the analytic function f may be written

$$f(z) = \sum_{n=2}^{\infty} \alpha_n z^n$$

then

$$S = A + \sum_{k=2}^{\infty}\left(\sum_{n=2}^{\infty}\frac{\alpha_n}{k^n}\right) = A + \sum_{n=2}^{\infty}\alpha_n\left(\sum_{k=2}^{\infty}\frac{1}{k^n}\right)$$

hence (11) may be transformed to

$$S = A + \sum_{n=2}^{\infty}\alpha_n\left(\zeta(n) - 1\right). \tag{12}$$

Observe that

$$\zeta(n) - 1 \sim \frac{1}{2^n},$$

therefore the transformed series (12) has a geometric rate of convergence. This rate may be improved if a few terms of the original series are computed, this time the limit is given by

$$S = A + \sum_{k=2}^{M} f\left(\frac{1}{k}\right) + \sum_{k=M+1}^{\infty} f\left(\frac{1}{k}\right) = A + \sum_{k=2}^{M} f\left(\frac{1}{k}\right) + \sum_{n=2}^{\infty}\alpha_n\left(\sum_{k=M+1}^{\infty}\frac{1}{k^n}\right)$$

and again

$$S = A + f\left(\frac{1}{2}\right) + \ldots + f\left(\frac{1}{M}\right) + \sum_{n=2}^{\infty}\alpha_n\left(\zeta(n) - 1 - \frac{1}{2^n} - \ldots - \frac{1}{M^n}\right)$$

but this time

$$\zeta(n) - 1 - \frac{1}{2^n} - \ldots - \frac{1}{M^n} = \zeta(n, M+1) \sim \frac{1}{(M+1)^n}$$

and $\zeta(s, a)$ is the *Hurwitz Zeta Function.* The rate of convergence remains geometric but this rate may be taken as large as desired by taking a large enough value for M.

7.1 Examples

1. The first natural example comes with the (almost) definition series for *Euler's constant*

$$S = \gamma = 1 + \sum_{k=2}^{\infty} \left(\frac{1}{k} + \log\left(1 - \frac{1}{k}\right) \right)$$

   here

$$f(z) = z + \log(1 - z) = -\sum_{n=2}^{\infty} \frac{z^n}{n}$$

   and the transformed series is

$$\gamma = 1 - \sum_{n=2}^{\infty} \frac{(\zeta(n) - 1)}{n}.$$

2. Another interesting series is *Mercator's relation* :

$$S = \log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \frac{1}{2} + \sum_{k=2}^{\infty} \frac{1}{(2k-1)2k}$$

   and this time

$$f(z) = \frac{z^2}{2(2-z)} = \sum_{n=2}^{\infty} \frac{z^n}{2^n}$$

   giving

$$\log(2) = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{(\zeta(n) - 1)}{2^n}$$

   and if we compute two more terms

$$\log(2) = \frac{37}{60} + \sum_{n=2}^{\infty} \frac{(\zeta(n) - 1 - 1/2^n - 1/3^n)}{2^n}.$$

3. A check relation

   Observe that if we take for the function f , $\alpha_n = 1$ for every n

$$f(z) = \sum_{n=2}^{\infty} z^n = \frac{1}{1-z} - 1 - z = \frac{z^2}{1-z}$$

   so that for $k > 1$

$$f\left(\frac{1}{k}\right) = \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$$

   and because clearly

$$S = \sum_{k=2}^{\infty} f\left(\frac{1}{k}\right) = \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1,$$

   we find the relation

$$\sum_{n \geq 2} (\zeta(n) - 1) = 1.$$

   With the same method come the two other relations

$$\begin{aligned} \sum_{n \geq 1} (\zeta(2n) - 1) &= \frac{3}{4} \\ \sum_{n \geq 1} (\zeta(2n+1) - 1) &= \frac{1}{4} \end{aligned}$$

   which may be used to check the evaluations of the Zeta function on consecutive integer values.

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