Distribution of the zeros of the Riemann Zeta function

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One of the most celebrated problem of mathematics is the *Riemann hypothesis* which states that all the non trivial zeros of the Zeta-function lie on the critical line $\Re(s) = 1/2$. Even if this famous problem is unsolved for so long, a lot of things are known about the zeros of $\zeta(s)$ and we expose here the most classical related results : all the non trivial zeros lie in the critical strip, the number of such zeros with ordinate less than T is proportional to $T \log T$, most zeros concentrate along the critical line $\sigma = 1/2$, there exists an infinity of zeros on the critical line and moreover, more than two fifth of the zeros are on the critical line.

1 Generalities

Let us recall (see section on the analytic continuation of $\zeta(s)$ in *The Riemann Zeta-function : generalities*) that zeta vanishes at negative odd integers. These zeros are called the *trivial zeros* of $\zeta(s)$. The functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \qquad \chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s). \tag{1}$$

entails that other zeros of $\zeta(s)$ (they are called the *non trivial zeros*) are symetric with respect to the critical line $\Re(s) = 1/2$: for each non trivial zero $s = \sigma + it$, the value $s' = 1 - \sigma + it$ is also a zero of $\zeta(s)$.

The non trivial zeros lie in the critical strip

We show that all the non trivial zeros of $\zeta(s)$ lie in the *critical strip* defined by values of the complex number s such that $0 < \Re(s) < 1$. Because of the functional equation, it suffices to show that $\zeta(s)$ does not vanish on the closed half plane $\Re(s) \geq 1$.

The Euler infinite product (see *The Riemann Zeta-function : generalities*)

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

valid for all complex numbers s with $\Re(s) > 1$, shows that $\zeta(s)$ does not vanish for $\Re(s) > 1$ (a convergent infinite product can not converge to zero because its logarithm is a convergent series). Thus it suffices now to prove that $\zeta(s)$

[1]This pages are from *//numbers.computation.free.fr/Constants/constants.html*

does not vanish on the line $\Re(s) = 1$. This property is in fact the key in the proof of the prime number theorem, and Hadamard and De La Vallée Poussin obtained this result independantly in 1896 by different mean (this problem is in fact a first step in a determination of a zero-free region, important to obtain good error terms in the prime number theorem). We present here the argument of De La Vallée Poussin which is simpler to expose and more elegant, in a form close to the presentation of [3].

The Zeta-function has no zeros on the line $\Re(s) = 1$

The starting point is the relation

$$3 + 4\cos\phi + \cos 2\phi = 2(1 + \cos\phi)^2 \geq 0 \tag{2}$$

for all values of the real number ϕ. The Euler infinite product writes as

$$\zeta(s) = \exp\left(\sum_{p\ \text{prime}} \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}\right), \qquad \Re(s) > 1$$

thus for the complex number $s = \sigma + it$ we have

$$|\zeta(s)| = \exp\left(\sum_{p\ \text{prime}} \sum_{m=1}^{\infty} \frac{\cos(mt\log p)}{mp^{m\sigma}}\right), \qquad \sigma = \Re(s) > 1.$$

This relation entails

$$\zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| = \exp\left(\sum_{p\ \text{prime}} \sum_{m=1}^{\infty} \frac{3 + 4\cos(mt\log p) + \cos(2mt\log p)}{mp^{m\sigma}}\right)$$

so with (2) we deduce

$$\zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \geq 1, \qquad \sigma > 1. \tag{3}$$

Now suppose that $1+it$ is a zero of $\zeta(s)$. Letting $\sigma \to 1$, we have $\zeta(\sigma) \sim 1/(\sigma-1)$ and $\zeta(\sigma+it) = O(\sigma-1)$, so that $\zeta(\sigma)^3|\zeta(\sigma+it)|^4 = O(\sigma-1)$ thus as $\sigma \to 1$, (3) entails that $|\zeta(\sigma+2it)|$ tends to infinity, which is impossible since $\zeta(s)$ is analytic around $1+2it$. Thus we have proved that $\zeta(s)$ has no zeros on the line $\Re(s) = 1$.

Other proofs of this result can be found in [3].

2 Counting the number of zeros in the critical strip

Even if precise location of zeros are not known, a lot of results have been obtained on counting the number of zeros in the critical strip. All the results in this section are more proved and detailed in [3, Ch. 9].

2.1 The number of zeros less than a given height

We let

$$\begin{aligned} \theta(t) &= \arg\left(\pi^{-it/2}\Gamma(1/4+it/2)\right) \\ S(t) &= \frac{1}{\pi}\arg\zeta(1/2+it) \end{aligned}$$

where the arguments are defined by continuous variation of s starting with the value 0 at $s=2$, going up vertically to $s=2+it$ and then horizontally to $s=\frac{1}{2}+it$ (when there is a zero of Zeta on the segment between $\frac{1}{2}+it$ and $2+it$, $S(T)$ is defined by $S(T+0)$). We already encountered the $\theta(t)$ function in *Numerical evaluation of the Riemann Zeta-function* while defining the Riemann-Siegel function $Z(t)$. If $N(T)$ denotes the number of zeros of $\zeta(\sigma+it)$ in the region $0<\sigma<1$, $0<t\le T$, then a result in[3] states that

$$N(T) = 1 + \frac{\theta(T)}{\pi} + S(T),$$

and using the asymptotic expansion of $\theta(T)$ (see *Numerical evaluation of the Riemann Zeta-function*) we have

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right). \tag{4}$$

The function $S(T)$ satisfies

$$S(T) = O(\log T), \qquad T \to +\infty, \tag{5}$$

which permits to say that $N(T)$ behaves like $T/(2\pi)\log(T/(2\pi))$. So the zeros of the Zeta-function become more and more dense as one goes up in the critical strip. More precisely the height of the n-th zero (ordered in increasing values of its ordinate) behaves like $2\pi n/\log n$. The results above also permit to state that the gap between the ordinates of successive zeros is bounded.

More on the function $S(T)$

The function $S(T)$ is important in local study of zeros and appears to be quite complicated. Today, no improvement of the bound (5) is known (under the Riemann hypothesis, we have the strongest bound $S(T)=O(\log T/\log\log T)$), but other results have been obtained. For example, we have the bound

$$\int_0^T S(t)dt = O(\log T), \tag{6}$$

which in particular, entails that the average value of $S(T)$ is zero. It is also known that $S(T)$ is not too small, and more precisely, a result from Selberg states that there exists a constant $A>0$ for which the inequality

$$|S(T)| > A(\log T)^{1/3}(\log\log T)^{-7/3}$$

holds for an infinity of values of T tending to infinity. Another result from Selberg gives

$$\int_0^T S(t)^2\,dt \sim \frac{1}{2\pi^2} T \log\log T,$$

thus the average value of $S(t)^2$ on $[0,T]$ is $1/(2\pi^2)\log\log T$ (this result has latter been refined by Ghosh in 1983 who proved that $|S(T)|/(\log\log T)^{1/2}$ has a limiting distribution). So $S(t)$ is, in average, tending to infinity very very slowly.

Finally a result of Titchmarsh states that the function $S(T)$ changes its sign infinitly often.

2.2 The zeros off the critical line

Even if Riemann hypothesis has not been proved, it is known that the zeros of the Zeta-function concentrate along the critical line. To precise this result, we define, when $0 < \sigma < 1/2$, the function $N(\sigma, T)$ as the number of zeros $s = \beta + it$ of Zeta such that $\beta > \alpha$ and $0 < t \le T$ (under the Riemann hypothesis one has $N(1/2, T) = 0$). A first result about $N(\sigma, T)$ states that for any fixed $\sigma > 1/2$, one has

$$N(\sigma, T) = O(T). \tag{7}$$

Since $N(T) \sim T/(2\pi)\log(T/(2\pi))$, we deduce that for any $\delta > 0$, all but an infinitesimal proportion of the zeros lie in the strip $1/2 - \delta < \sigma < 1/2 + \delta$.

There exists numerous results that improve the bound (7) in some context, for example we have

$$\begin{aligned} N(\sigma,T) &= O(T^{4\sigma(1-\sigma)+\epsilon}), \quad \text{forall } \epsilon > 0, \\ N(\sigma,T) &= O(T^{3/2-\sigma}\log^5 T) \\ N(\sigma,T) &= O(T^{3(1-\sigma)/(2-\sigma)}\log^5 T). \end{aligned}$$

A result of a different kind has been obtained by Selberg, who proved

$$\int_{1/2}^1 N(\sigma, T)d\sigma = O(T).$$

3 The zeros on the critical line

The Riemann hypothesis states that all non trivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = 1/2$. Even if this has never been proved or disproved, mathematicians succeeded in proving that there exists an infinity of non trivial zeros on the critical line, and later, that a positive proportion of the zeros are on the line.

3.1 An infinity of zeros are on the critical line

Hardy was the first to prove in 1914 that an infinity of zeros are on the critical line. Later, other mathematicians like Pólya in 1927, Landau, or Titchmarsh in the 1930's gave other proofs. In [3], five different proofs are given.

We sketch here the ideas under the Titchmarsh approach, which makes use the Riemann-Siegel Z function (see *The Riemann-Siegel Z-function* in *Numerical evaluation of the Riemann Zeta-function*), with considerations close to what is used in numerical computations of the zeros. We recall that the $Z(t)$ function is purely real and satisfies $|Z(t)| = |\zeta(1/2+it)|$, and that Riemann-Siegel formula for $Z(t)$ writes as

$$Z(t) = 2\sum_{1\le n\le x} \frac{\cos(\theta(t) - t\log n)}{\sqrt{n}} + O(t^{-1/4}), \qquad x = \sqrt{\frac{t}{2\pi}}. \tag{8}$$

Titchmarsh observed that the dominant term in the sum above is obtained with $n = 1$ and is $\cos\theta(t)$ and thus one should expect that on average, the sign of $Z(t)$ would be the sign of $\cos\theta(t)$. For that reason he defined t_ν as the solution of $\theta(t_\nu) = \nu\pi$ (we have $t_\nu \sim 2\pi\nu/\log\nu$) and showed that on average, the value of $Z(t_\nu)$ is $2(-1)^\nu$. More precisely, he proved that

$$\sum_{\nu\le N} Z(t_{2\nu}) \sim 2N, \qquad \sum_{\nu\le N} Z(t_{2\nu+1}) \sim -2N \tag{9}$$

by showing that in the ν summation, the terms $\cos(\theta(t_{2\nu}) - t_{2\nu}\log n)n^{-1/2}$ and $\cos(\theta(t_{2\nu+1}) - t_{2\nu+1}\log n)n^{-1/2}$ of (8) for $n \ge 2$ cancellate and give a contribution of inferior order. It is now easy to prove that $Z(t)$ has an infinity of zeros since if not, it would keep the same sign for t large enough, thus one of the two estimates in (9) would not be satisfied.

3.2 A finite proportion of zeros lie on the critical line

After Hardy proof that there exists an infinity of zeros on the critical line, strongest results that estimate the minimal number of zeros on the line have been obtained. Denoting by $N_0(T)$ the number of zeros of $\zeta(1/2+it)$ for $0 < t \le T$, the first strongest historical result is due to Hardy and Littlewood who proved in 1921 that there exists a constant $A > 0$ for which the inequality

$$N_0(T) > AT$$

holds for all values of T (see [3] for a proof). Selberg in 1932 improved considerably this result by showing the existence of a constant $A > 0$ for which

$$N_0(T) > AT\log T$$

holds (the numerical value of A is Selberg proof is very small). Since $N(T) \sim T/(2\pi)\log T$, it means that a finite proportion of the zeros lie on the critical line. The proof is quite complicated and is given in [3]. The most significant result on $N_0(T)$ has been obtained by Conrey in 1989 (see [1]) and states that for T large enough, one as

$$N_0(T) \geq \alpha N(T), \qquad \alpha = 0.40219.$$

Thus more than one third of the zeros lie on the critical line.

References

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[3] E. C. Titchmarsh, *The theory of the Riemann Zeta-function*, Oxford Science publications, second edition, revised by D. R. Heath-Brown (1986).