Collection of approximations for π

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Abstract

Selection of numerical approximations of the famous constant π.

1 Approximation formulae

The history of π is full of more or less good approximations.

1.1 Rational approximations

The first estimations of the ratio of the circumference to its diameter are found in the ancient times. The symbol π was not yet used to design this ratio.

$$
\begin{aligned}
\pi &\approx 3+\frac{1}{8}=3.1(25) \qquad \text{(Babylonians)} \\
\pi &\approx \left(\frac{4}{3}\right)^4=3.1(604...) \qquad \text{(Egyptians)} \\
\pi &\approx \frac{22}{7}=3.14(285...) \qquad \text{(Archimedes 287-212 B.C.)} \\
\pi &\approx 3+\frac{8}{60}+\frac{30}{60^2}=3.141(666...) \qquad \text{(C. Ptolemy 100-170)} \\
\pi &\approx \frac{62832}{20000}=3.141(6) \qquad \text{(India around 500)} \\
\pi &\approx \frac{333}{106}=3.1415(094...) \\
\pi &\approx \frac{355}{113}=3.141592(920...) \qquad \text{(Zu Chongzhi, Adriaen Métius, ...)}
\end{aligned}
$$

Ptolemy gave his approximate value of π in his *Almagest* and he used sexagesimal fractions [1].

The famous and remarquable value 355/113 was published in 1625 by Adriaen Metius but it was already used in China around 480 and also known to the Japanese.

1.1.1 Continued fractions

The theory of continued fractions provides the sequence of the best rational approximations for the number π, the first terms are given recursively by (starting with $x = \pi$ and $n = 0$)

$$\begin{aligned} i_n &= [x] \\ x &= \frac{1}{x - i_n} \\ n &= n + 1. \end{aligned}$$

It follows that $i_0 = 3, i_1 = 7, i_2 = 15, ...$ and it's usual to write it as :

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, ...]$$

which becomes in term of fractions

$$\left[3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \frac{312689}{99532}, \frac{833719}{265381}, \frac{1146408}{364913}, \frac{4272943}{1360120}, ...\right]$$

and for example the fraction

$$\frac{428224593349304}{136308121570117} = 3.14159265358979323846264338327(569...)$$

has 29 correct digits and it was found by the the Japenese mathematician Arima in 1766. We know from an important theorem of continued fraction (see [2] for a proof) that the error of the approximation p_n/q_n satisfies

$$\left|\frac{p_n}{q_n} - \pi\right| \leq \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}.$$

1.2 With square roots

$$\begin{aligned}
\pi &\approx \sqrt{10} = 3.1(622...) \qquad \text{(India around 600)} \\
\pi &\approx \frac{3}{4}\left(\sqrt{3} + \sqrt{6}\right) = 3.1(361...) \qquad \text{(Nicolaus de Cusa-1464)} \\
\pi &\approx \sqrt{2} + \sqrt{3} = 3.14(626...) \\
\pi &\approx \frac{88}{\sqrt{785}} = 3.14(085...) \qquad \text{(Tycho Brahe-1580)} \\
\pi &\approx 10\sqrt{2} - 11 = 3.14(213...) \qquad \text{(Grosvenor-1868)} \\
\pi &\approx \sqrt{51} - 4 = 3.141(428...) \\
\pi &\approx \sqrt{\frac{40}{3} - 2\sqrt{3}} = 3.1415(333...) \qquad \text{(Kochansky-1685)} \\
\pi &\approx \frac{13}{50}\sqrt{146} = 3.14159(195...) \qquad \text{(Specht-1836)} \\
\pi &\approx \frac{141}{1232}\left(5 + 6\sqrt{14}\right) = 3.14159265358(015...)
\end{aligned}$$

Kochansky's value (Poland) was published in 1685 and is the result of a geometrical construction for π. Another construction was given by Specht (Germany) for it's value.

1.3 Ramanujan's approximations

In his work on modular equations Ramanujan found impressive approximations, this is just a small selection [4].

$$\begin{aligned}
\pi &\approx \left(9^2+\frac{19^2}{22}\right)^{1/4} = 3.14159265(258...)\\
\pi &\approx \frac{9}{5}+\sqrt{\frac{9}{5}} = 3.141(640...)\\
\pi &\approx \frac{9801}{4412}\sqrt{2} = 3.141592(730...)\\
\pi &\approx \frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) = 3.141592653(805...)\\
\pi &\approx \frac{12}{\sqrt{130}}\log\left(\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right) = 3.14159265358979(265...)\\
\pi &\approx \frac{12}{\sqrt{190}}\log\left((2\sqrt{2}+\sqrt{10})(3+\sqrt{10})\right) = 3.141592653589793238(190...)
\end{aligned}$$

The last approximations are of the form

$$\pi \approx \frac{2}{\sqrt{n}}\log\left(8g_n^{12}\right)$$

where usually n is an integer and g_n a Ramanujan's invariant.

To conclude this section here is a remarkable approximation of π published in 1984 by Morris Newman and Daniel Shanks [3]. Set

$$\begin{aligned}
a &= \frac{1071}{2}+92\sqrt{34}+\frac{3}{2}\sqrt{255349+43792\sqrt{34}},\\
b &= \frac{1553}{2}+133\sqrt{34}+\frac{1}{2}\sqrt{4817509+826196\sqrt{34}},\\
c &= 429+304\sqrt{2}+2\sqrt{92218+65208\sqrt{2}},\\
d &= \frac{627}{2}+221\sqrt{2}+\frac{1}{2}\sqrt{783853+554268\sqrt{2}},
\end{aligned}$$

then

$$\left|\pi - \frac{6}{\sqrt{3502}}\log(2abcd)\right| < 7.4.10^{-82}.$$

1.4 Other approximations

$$
\begin{aligned}
\pi &\approx \left(\frac{2143}{22}\right)^{1/4} = 3.14159265(258...) \qquad \text{(Plouffe)} \\
\pi &\approx \left(\frac{77729}{254}\right)^{1/5} = 3.14159265(411...) \qquad \text{(Castellanos)} \\
\pi &\approx \frac{6}{5}\log\left(7+3\sqrt{5}\right) = \frac{6}{5}\log(2) + \frac{24}{5}\log(\varphi) = 3.14159(337...) \\
\pi &\approx \frac{e^2}{3} + \frac{19}{28} = 3.14159(012...) \\
\pi &\approx \frac{4}{\sqrt{\varphi}} = 3.14(460...) \qquad \text{(Ghyka)} \\
\pi &\approx \frac{6}{5}(1+\varphi) = 3.141(640...)
\end{aligned}
$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}$$

is the *Golden ratio*.

References

[1] F. Cajori, *A History of Mathematical notations*, Dover, (republication 1993, original 1928-1929)

[2] G.H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford Science Publications, (1979)

[3] M. Newman, D. Shanks, *On a Sequence Arising in Series for* π, Math. of Comp., (1984), vol. 42, p. 199-217

[4] S. Ramanujan, *Modular equations and approximations to* π, Quart. J. Pure Appl. Math., (1914), vol. 45, p. 350-372