π and its computation through the ages

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The value of π has engaged the attention of many mathematicians and calculators from the time of Archimedes to the present day, and has been computed from so many different formulae, that a complete account of its calculation would almost amount to a history of mathematics.

- James Glaisher (1848-1928)

The history of pi is a quaint little mirror of the history of man.

- Petr Beckmann

1 Computing the constant π

Understanding the nature of the constant π, as well as trying to estimate its value to more and more decimal places has engaged a phenomenal energy from mathematicians from all periods of history and from most civilizations. In this small overview, we have tried to collect as many as possible major calculations of the most famous mathematical constant, including the methods used and references whenever there are available.

1.1 Milestones of π's computation

- Ancient civilizations like Egyptians [15], Babylonians, China ([29], [52]), India [36],... were interested in evaluating, for example, area or perimeter of circular fields. Of course in this early history, π was not yet a constant and was only *implicit* in all available documents.

  Perhaps the most famous is the *Rhind Papyrus* which states the rule used to compute the area of a circle: *take away 1/9 of the diameter and take the square of the remainder* therefore implicitly $\pi = (16/9)^2$.

- Archimedes of Syracuse (287-212 B.C.). He developed a method based on inscribed and circumscribed *polygons* which will be of practical use until the mid seventeenth. It is the first known algorithm to compute π to, in principle, any required accuracy. By mean of regular polygons with 96 sides [22], in his treatise *Measurement of a Circle,* he showed that: *the ratio of the circumference of any circle to its diameter is less than* $3+1/7$ *but greater than* $3+10/71$.

- Zu Chongzhi from China (430-501). He established that $3.1415926 < \pi < 3.1415927$ using polygons [29].

- Al-Kashi from Samarkand (1380-1429). By mean of Archimedes' polygons, he computed, in 1424, 2π to nine sexagesimal places and his estimation (about 14 correct decimals) will remain unsurpassed for nearly 200 years [1].

- Ludolph van Ceulen (1540-1610). Still using polygons, he made various computations and finally found 35 decimal places before his death in 1610, [10]. On his tombstone (today lost [23]), the following approximation was engraved: *when the diameter is 1, then the circumference of the circle is greater than*

$$\frac{314159265358979323846264338327950288}{100000000000000000000000000000000000}$$

  but smaller than

$$\frac{314159265358979323846264338327950289}{100000000000000000000000000000000000}.$$

- Isaac Newton (1643-1727) and James Gregory (1638-1675) introduced respectively the series expansion of the function arcsin (1669) and the function arctan (1671) and opened the era of *analytical methods* to compute π. After a 15 digits computation Newton wrote: *I am ashamed to tell you to how many figures I carried these computations, having no other business at the time* [6]*.*

- John Machin (1680-1751). In 1706 [25], *the Truly Ingenious Mr. John Machin* reached 100 decimal places with a fast converging arctan formula which now bares his name. The same year, William Jones (1675-1749) uses the symbol π to represent the ratio of the perimeter of a circle to its diameter.

- Johann Heinrich Lambert (1728-1777). π is *irrational* (1761, [30]).

- Adrien Marie Legendre (1752-1833). π^2 is *irrational* (1794, [31]).

- William Shanks (1812-1882). He spent a considerable part of his life to compute various approximations of π including a final 707 digits estimation ([42], [43]); this performance remains probably the most impressive of this nature. It was not until 1946 that an unfortunate mistake was discovered at the 528th place [16].

- Carl Louis Ferdinand von Lindemann (1852-1939). π is *transcendental* (1882, [33]).

- François Genuys, Daniel Shanks with John Wrench and Jean Guilloud with Martine Bouyer respectively reached 10,000 (1958, [19]), 100,000 (1961, [44]) and 1,000,000 (1973, [21]) digits by using arctan formulae and *classical series expansion* computation.

- Richard Brent and Eugene Salamin. They published in 1976, two important articles ([9], [40]) describing a new *iterative* and *quadratic* algorithm to determine π. This opened the era of *fast algorithms*, that is algorithms with complexity nearly proportional to the number of computed decimal places.

  Other methods of this nature and with higher order of convergence were later developed by Peter and Jonathan Borwein [7].

- David Chudnovsky and Gregory Chudnovsky. Introduction of new very fast series (consecutive to Ramanujan's work, [37], [12]) to establish various record on a home made supercomputer *m-zero*! The first billion digits was achieved by them in 1989 (see [35]).

- Yasumada Kanada. Since 1980 he is one of the major actor in the race to compute π to huge number of digits. Most of his calculations are made on supercomputers and are based on modern high order iterative algorithms (see [48], [26], [27], [47],...)

- Fabrice Bellard. At the end of 2009, nearly 2700 billion decimal digits of π were computed on a single desktop computer and it took a total of 131 days to achieve this performance.

Other enumerations of π's computations can also be found in: [2], [3], [6], [14], [41], [54],... in which we got many valuable informations.

The two following sections are enumerating the main computations respectively before and during computer era. The *Method* column usually refers to the last section; for example arctan(M) means that the arctangent relation (M) or *Machin's formula* was used.

2 History of π calculations before computer era

Author	Year	Exact digits	Method	Comment
Egyptians	2000 B.C.	1	unknown	$\pi = (16/9)^2$
Babylonians	2000 B.C.	1		$\pi = 3 + 1/8$
Bible	550 B.C. ?	0		$\pi = 3$
Archimedes, [22]	250 B.C.	2	polygon	$\pi = 22/7$, 96 sides
Ptolemy	150	3		$\pi = 3 + 8/60 + 30/60^2$
Liu Hui, [29]	263	5	polygon	3072 sides
Zu Chongzhi, [29]	480	7	polygon	Also $\pi = 355/113$
Aryabhata	499	4	polygon	$\pi = 62832/20000$
Brahmagupta	640	1		$\pi = \sqrt{10}$
Al-Khwarizmi	830	4		$\pi = 62832/20000$
Fibonacci	1220	3	polygon	$\pi = 3.141818$
Al-Kashi, [1]	1424	14	polygon	6.2^{27} sides
Otho	1573	6		$\pi = 355/113$
Viète	1579	9	polygon	6.2^{16} sides
van Roomen	1593	15	polygon	2^{30} sides
van Ceulen	1596	20	polygon	60.2^{33}sides
van Ceulen, [10]	1610	35	polygon	2^{62} sides
van Roijen Snell, [45]	1621	34	polygon	2^{30} sides
Grienberger	1630	39	polygon	
Newton	1671	15	series(N)	
Sharp	1699	71	arctan(Sh)	
Machin, [25]	1706	100	arctan(M)	
De Lagny, [28]	1719	112	arctan(Sh)	127 computed
Takebe Kenko	1722	40	series	
Matsunaga	1739	49	series	
Euler	1755	20	arctan(E2)	In one hour!
Vega	1789	126	arctan(H)	143 computed
Vega, [51]	1794	136	arctan(H)	140 computed
Rutherford, [39]	1841	152	arctan(E1)	208 computed
Dahse, [13]	1844	200	arctan(SD)	
Clausen	1847	248	arctan(H & M)	
Lehmann	1853	261	arctan(E3)	
Shanks, [42]	1853	527	arctan(M)	607 computed
Rutherford	1853	440	arctan(M)	
Richter	1854	500	unknown	
Shanks, [43]	1873	527	arctan(M)	707 computed
Tseng Chi-hung	1877	100	arctan(E3)	
Uhler	1900	282	arctan(M)	
Duarte	1902	200	arctan(M)	
Uhler, [50]	1940	333		
Ferguson	1944-1945	530	arctan(L)	
Ferguson, [16]	07-1946	620	arctan(L)	Last hand calculation

3 History of π calculations during computer era

Author	**Year**	**Exact digits**	**Method**	**Computer**
Ferguson	01-1947	710	arctan(L)	
Ferguson & Wrench Jr	09-1947	808	arctan(M)	
Smith & Wrench Jr, [53]	06-1949	1,120	arctan(M)	
Reitwiesner et al., [38]	09-1949	2,037	arctan(M)	ENIAC
Nicholson & Jeenel, [34]	11-1954	3,092	arctan(M)	NORC
Felton	03-1957	7,480	arctan(K & G)	Pegasus
Genuys, [19]	01-1958	10,000	arctan(M)	IBM 704
Felton	05-1958	10,020	arctan(K & G)	Pegasus
Guilloud	07-1959	16,167	arctan(M)	IBM 704
Shanks & Wrench Jr, [44]	07-1961	100,265	arctan(S1 & G)	IBM 7090
Guilloud & Filliatre	02-1966	250,000	arctan(S1 & G)	IBM 7030
Guilloud & Dichampt	02-1967	500,000	arctan(S1 & G)	CDC 6600
Guilloud & Bouyer, [21]	05-1973	1,001,250	arctan(S1 & G)	CDC 7600
Kanada & Miyoshi	1981	2,000,036	arctan(K & M)	FACOM M-200
Guilloud	1982	2,000,050	unknown	unknown
Tamura	1982	2,097,144	GL2	MELCOM 900II
Tamura & Kanada, [48]	1982	4,194,288	GL2	Hitachi M-280H
Tamura & Kanada	1982	8,388,576	GL2	Hitachi M-280H
Kanada et al.	1983	16,777,206	GL2	Hitachi M-280H
Kanada et al., [26]	10-1983	10,013,395	arctan(G), GL2	Hitachi S-810/20
Gosper	10-1985	17,526,200	series(Ra), B4	Symbolics 3670
Bailey, [4]	01-1986	29,360,111	B2, B4	CRAY-2
Kanada & Tamura	09-1986	33,554,414	GL2, B4	Hitachi S-810/20
Kanada & Tamura	10-1986	67,108,839	GL2	Hitachi S-810/20
Kanada et al.	01-1987	134,214,700	GL2, B4	NEC SX-2
Kanada & Tamura, [27]	01-1988	201,326,551	GL2, B4	Hitachi S-820/80
Chudnovskys	05-1989	480,000,000	series	CRAY-2
Chudnovskys	06-1989	525,229,270	series	IBM 3090
Kanada & Tamura	07-1989	536,870,898	GL2	Hitachi S-820/80
Chudnovskys	08-1989	1,011,196,691	series(CH)	IBM 3090 & CRAY-2
Kanada & Tamura	11-1989	1,073,741,799	GL2, B4	Hitachi S-820/80
Chudnovskys, [35]	08-1991	2,260,000,000	series(CH?)	m-zero
Chudnovskys	05-1994	4,044,000,000	series(CH)	m-zero
Kanada & Takahashi	06-1995	3,221,220,000	GL2, B4	Hitachi S-3800/480
Kanada & Takahashi	08-1995	4,294,967,286	GL2, B4	Hitachi S-3800/480
Kanada & Takahashi	10-1995	6,442,450,000	GL2, B4	Hitachi S-3800/480
Chudnovskys	03-1996	8,000,000,000	series(CH?)	m-zero ?
Kanada & Takahashi	04-1997	17,179,869,142	GL2, B4	Hitachi SR2201
Kanada & Takahashi, [47]	06-1997	51,539,600,000	GL2, B4	Hitachi SR2201
Kanada & Takahashi	04-1999	68,719,470,000	GL2, B4	Hitachi SR8000
Kanada & Takahashi	09-1999	206,158,430,000	GL2, B4	Hitachi SR8000
Kanada et al.	12-2002	1,241,100,000,000	arctan(S2 & S3)	Hitachi SR8000/MP
Daisuke et al.	08-2009	2,576,980,370,000	GL2, B4	T2K Supercomputer
Bellard	12-2009	2,699,999,990,000	series(CH)	PC Intel core i7
Yee & Kondo	08-2010	5,000,000,000,000	series(CH)	PC Intel Xeon

4 List of the main used methods

In this section are expressed the main identities used to compute π just after the geometric period which was based on the computation of the perimeter (or area) of regular polygons with many sides.

4.1 Machin like formulae

There are numerous more or less efficient formulae to compute π by mean of arctan functions (see [3], [24], [32], [46], [49],...).

$$\frac{\pi}{6} = \arctan \frac{1}{\sqrt{3}} \tag{Sh}$$

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}, \text{ Machin} \tag{M}$$

$$\frac{\pi}{4} = 8 \arctan \frac{1}{10} - \arctan \frac{1}{239} - 4 \arctan \frac{1}{515}, \text{ Klingenstierna} \tag{K}$$

$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{8}, \text{ Strassnitzky} \tag{SD}$$

$$\frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}, \text{ Gauss} \tag{G}$$

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{70} + \arctan \frac{1}{99}, \text{ Euler} \tag{E1}$$

$$\frac{\pi}{4} = 5 \arctan \frac{1}{7} + 2 \arctan \frac{3}{79}, \text{ Euler} \tag{E2}$$

$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}, \text{ Euler} \tag{E3}$$

$$\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}, \text{ Hutton} \tag{H}$$

$$\frac{\pi}{4} = 3 \arctan \frac{1}{4} + \arctan \frac{1}{20} + \arctan \frac{1}{1985}, \text{ Loney} \tag{L}$$

$$\frac{\pi}{4} = 6 \arctan \frac{1}{8} + 2 \arctan \frac{1}{57} + \arctan \frac{1}{239}, \text{ Störmer} \tag{S1}$$

$$\frac{\pi}{4} = 12 \arctan \frac{1}{49} + 32 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} + 12 \arctan \frac{1}{110443} \tag{S2}$$

$$\frac{\pi}{4} = 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12943} \tag{S3}$$

4.2 Other series

4.2.1 Newton

$$
\begin{aligned}
\pi &= \frac{3\sqrt{3}}{4} + 24\int_0^{1/4} \sqrt{x - x^2}dx \\
&= \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{5.2^5} - \frac{1}{28.2^7} - \frac{1}{72.2^9} - \cdots\right), \text{ Newton} \qquad \text{(N)}
\end{aligned}
$$

4.2.2 Ramanujan like series

The important point is that evaluating such series to huge number of digits requires to develop specific algorithms. Such algorithms are now well known and are based on idea related to *binary splitting*. To learn more about those consult: [7], [8], [20],...

$$
\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!}{(k!)^4 4^{4k}}\frac{(1103 + 26390k)}{99^{4k}}, \quad \text{Ramanujan [37]} \qquad \text{(Ra)}
$$

$$
\frac{1}{\pi} = 12\sum_{k=0}^{\infty}(-1)^k\frac{(6k)!}{(3k)!(k!)^3}\frac{(13591409 + 545140134k)}{640320^{3k+3/2}}, \text{ Chudnovsky} \qquad \text{(CH)}
$$

4.3 Iterative algorithms

The main difficulty with the following iterative procedures is to compute to a high accuracy inverses and square roots of a real number. By mean of *FFT* based methods to compute products of numbers with many decimal places this is now possible in a quite efficient way. To find how to compute those operations you can consult [3], [7], [9], [20],...

4.3.1 Gauss-Legendre (or Brent-Salamin) quadratic

Set $x_0 = 1, y_0 = 1/\sqrt{2}, \alpha_0 = 1/2$ and:

$$
\begin{cases}
x_{k+1} = (x_k + y_k)/2 \\
y_{k+1} = \sqrt{x_k y_k} \\
\alpha_{k+1} = \alpha_k - 2^{k+1}\left(x_{k+1}^2 - y_{k+1}^2\right)
\end{cases}
$$

then ([7], [9], [40]):

$$
\pi = \lim_{k\to\infty}\left(2x_k^2/\alpha_k\right). \qquad \text{(GL2)}
$$

4.3.2 Borwein quadratic

Set $x_0 = \sqrt{2}, y_0 = 0, \alpha_0 = 2 + \sqrt{2}$ and:

$$\begin{cases} x_{k+1} = \left(\sqrt{x_k} + 1/\sqrt{x_k}\right)/2 \\ y_{k+1} = \sqrt{x_k}\left(\frac{1+y_k}{y_k+x_k}\right) \\ \alpha_{k+1} = \alpha_k y_{k+1}\left(\frac{1+x_{k+1}}{1+y_{k+1}}\right) \end{cases}$$

then ([7]):

$$\pi = \lim_{k\to\infty} \alpha_k. \tag{B2}$$

4.3.3 Borwein quartic

Set $y_0 = \sqrt{2} - 1, \alpha_0 = 6 - 4\sqrt{2}$ and:

$$\begin{cases} y_{k+1} = \left(1 - (1 - y_k^4)^{1/4}\right)/(1 + (1 - y_k^4)^{1/4}) \\ \alpha_{k+1} = (1 + y_{k+1})^4\alpha_k - 2^{2k+3}y_{k+1}(1 + y_{k+1} + y_{k+1}^2) \end{cases}$$

then ([7]):

$$\pi = \lim_{k\to\infty} (1/\alpha_k). \tag{B4}$$

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