Collection of series for π

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numbers.computation.free.fr/Constants/constants.html

March 31, 2004

Abstract

Selection of some of the numerous series expansion involving the famous constant π.

1 Introduction

There are a great many numbers of series involving the constant π, we provide a selection. The celebrated Swiss mathematician Leonhard Euler (1707-1783) discovered many of those.

2 Around Leibniz-Gregory-Madhava series

$$\begin{aligned}
\frac{\pi}{4} &= 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots \qquad \text{(Leibniz-Gregory-Madhava)} \\
\frac{\pi^2}{16} &= \sum_{k\geq 0}\frac{(-1)^k}{k+1}\left(1+\frac{1}{3}+\cdots+\frac{1}{2k+1}\right) \qquad \text{(Knopp)} \\
\frac{\pi}{4} &= \frac{3}{4}+\frac{1}{2.3.4}-\frac{1}{4.5.6}+\frac{1}{6.7.8}-\cdots \qquad \text{(Nilakantha)} \\
\frac{\pi}{2} &= 1+\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\cdots \qquad \text{(Euler)} \\
\frac{\pi}{2} &= \frac{1.2}{1.3}+\frac{1.2.3}{1.3.5}+\frac{1.2.3.4}{1.3.5.7}+\cdots \\
\pi &= \sum_{k\geq 1}\frac{3^k-1}{4^k}\zeta(k+1) \qquad \text{(Flajolet-Vardi)} \\
\frac{\pi}{4} &= \sum_{k\geq 1}\arctan\frac{1}{k^2+k+1} \qquad \text{(Knopp)} \\
\frac{\pi}{4} &= \sum_{k\geq 1}\arctan\frac{1}{F_{2k+1}}
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{\pi} &= \sum_{k\geq 1} \frac{1}{2^{k+1}} \tan \frac{\pi}{2^{k+1}} \qquad \text{(Euler)} \\
\frac{\pi\sqrt{2}}{4} &= 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots \\
\frac{\pi\sqrt{3}}{6} &= 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \cdots \\
\frac{\pi\sqrt{3}}{9} &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots \\
\frac{\pi\sqrt{3}}{6} &= \sum_{k\geq 0} \frac{(-1)^k}{3^k(2k+1)} \qquad \text{(Sharp)}
\end{aligned}
$$

The F_n are Fibonacci numbers.

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5, ...,\ F_{n+1} = F_n + F_{n-1}$.

3 Euler's series

It was a great problem to find the limit of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \cdots,$$

some of the greatest mathematicians of the seventeenth century failed to find this limit. After trying unsuccessfully to solve it, Jakob Bernoulli challenged mathematicians with this problem. It was Euler, in 1735, who found the value of the series and most of the following result are also due to him. (See [4].)

3.1 All integers

$$
\begin{aligned}
\frac{\pi^2}{6} &= \sum_{k\geq 1} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \\
\frac{\pi^4}{90} &= \sum_{k\geq 1} \frac{1}{k^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots \\
\frac{\pi^6}{945} &= \sum_{k\geq 1} \frac{1}{k^6} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \cdots \\
\frac{4^p \left|B_{2p}\right| \pi^{2p}}{2(2p)!} &= \sum_{k\geq 1} \frac{1}{k^{2p}} = \zeta(2p)
\end{aligned}
$$

3.2 Odd integers

$$\begin{aligned}
\frac{\pi^2}{8} &= \sum_{k\geq 0} \frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \\
\frac{\pi^4}{96} &= \sum_{k\geq 0} \frac{1}{(2k+1)^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \\
\frac{\pi^6}{960} &= \sum_{k\geq 0} \frac{1}{(2k+1)^6} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \cdots \\
\frac{(4^p-1)\,|B_{2p}|\,\pi^{2p}}{2(2p)!} &= \sum_{k\geq 0} \frac{1}{(2k+1)^{2p}}
\end{aligned}$$

3.3 All integers alternating

$$\begin{aligned}
\frac{\pi^2}{12} &= \sum_{k\geq 1} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \\
\frac{7\pi^4}{720} &= \sum_{k\geq 1} \frac{(-1)^{k+1}}{k^4} = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \cdots \\
\frac{31\pi^6}{30240} &= \sum_{k\geq 1} \frac{(-1)^{k+1}}{k^6} = 1 - \frac{1}{2^6} + \frac{1}{3^6} - \cdots \\
\frac{(4^p-2)\,|B_{2p}|\,\pi^{2p}}{2(2p)!} &= \sum_{k\geq 1} \frac{(-1)^{k+1}}{k^{2p}}
\end{aligned}$$

3.4 Odd integers alternating

$$\begin{aligned}
\frac{\pi^3}{32} &= \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots \\
\frac{5\pi^5}{1536} &= \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)^5} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \cdots \\
\frac{61\pi^7}{184320} &= \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)^7} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \cdots \\
\frac{|E_{2p}|\,\pi^{2p+1}}{4^{p+1}(2p)!} &= \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)^{2p+1}}
\end{aligned}$$

B_n and E_n are respectively Bernoulli's numbers and Euler's numbers:

$$\begin{aligned} B_0 &= 1,\, B_1 = -\frac{1}{2},\, B_2 = \frac{1}{6},\, B_4 = -\frac{1}{30},\, B_6 = \frac{1}{42},\, B_8 = -\frac{1}{30},\, B_{10} = \frac{5}{66}, ..., \\ E_0 &= 1,\, E_2 = -1,\, E_4 = 5,\, E_6 = -61,\, E_8 = 1385,\, E_{10} = -50251, ... \end{aligned}$$

3.5 With prime numbers

In the following series, only the denominators with an odd number of prime factors are taken in account. For example the number $10 = 2 \times 5$ is omitted because it has two prime factors.

$$\begin{aligned} \frac{\pi^2}{20} &= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{11^2} + \cdots \\ \frac{\pi^4}{1260} &= \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{8^4} + \frac{1}{11^4} + \cdots \\ \frac{4\pi^6}{225225} &= \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{8^6} + \frac{1}{11^6} + \cdots \\ \frac{\zeta^2(2p) - \zeta(4p)}{2\zeta(2p)} &= \frac{1}{2^{2p}} + \frac{1}{3^{2p}} + \frac{1}{5^{2p}} + \frac{1}{7^{2p}} + \frac{1}{8^{2p}} + \frac{1}{11^{2p}} + \cdots \end{aligned}$$

If this time the prime factors are also supposed to be different:

$$\begin{aligned} \frac{9}{2\pi^2} &= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots \\ \frac{15}{2\pi^4} &= \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \frac{1}{13^4} + \cdots \\ \frac{11340}{691\pi^6} &= \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \frac{1}{13^6} + \cdots \\ \frac{\zeta^2(2p) - \zeta(4p)}{2\zeta(2p)\zeta(4p)} &= \frac{1}{2^{2p}} + \frac{1}{3^{2p}} + \frac{1}{5^{2p}} + \frac{1}{7^{2p}} + \frac{1}{11^{2p}} + \frac{1}{13^{2p}} + \cdots \end{aligned}$$

4 Machin's formulae

By mean of the function

$$\mathrm{L}(p) = \arctan \frac{1}{p} = \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)p^{2k+1}}$$

numerous more or less efficient formulae to express π are available. (Consult [1], [5], [7], [9].)

Observe that the *Leibniz-Gregory-Madhava* series may be written as $\frac{\pi}{4} = \mathrm{L}(1)$ and Sharp's series is equivalent to $\frac{\pi}{6} = \mathrm{L}(\sqrt{3})$.

4.1 Two terms formulae

$$\frac{\pi}{2} = 2\,\mathrm{L}(\sqrt{2}) + \mathrm{L}(2\sqrt{2}) \qquad \text{(Wetherfield)}$$
$$\frac{\pi}{4} = \mathrm{L}(2) + \mathrm{L}(3) \qquad \text{(Hutton)}$$
$$\frac{\pi}{4} = 2\,\mathrm{L}(3) + \mathrm{L}(7) \qquad \text{(Hutton)}$$
$$\frac{\pi}{4} = 4\,\mathrm{L}(5) - \mathrm{L}(239) \qquad \text{(Machin)}$$
$$\frac{\pi}{6} = 2\,\mathrm{L}(3\sqrt{3}) + \mathrm{L}(4\sqrt{3})$$
$$\frac{\pi}{4} = 5\,\mathrm{L}(7) + 2\,\mathrm{L}\left(\frac{79}{3}\right) \qquad \text{(Euler)}$$
$$\frac{\pi}{4} = 5\,\mathrm{L}\left(\frac{278}{29}\right) + 7\,\mathrm{L}\left(\frac{79}{3}\right)$$

4.2 Three terms and more formulae

$$\frac{\pi}{4} = \mathrm{L}(2) + \mathrm{L}(5) + \mathrm{L}(8) \qquad \text{(Strassnitzky)}$$
$$\frac{\pi}{4} = 4\,\mathrm{L}(5) - \mathrm{L}(70) + \mathrm{L}(99) \qquad \text{(Euler)}$$
$$\frac{\pi}{4} = 5\,\mathrm{L}(7) + 4\,\mathrm{L}(53) + 2\,\mathrm{L}(4443)$$
$$\frac{\pi}{4} = 6\,\mathrm{L}(8) + 2\,\mathrm{L}(57) + \mathrm{L}(239) \qquad \text{(Störmer)}$$
$$\frac{\pi}{4} = 8\,\mathrm{L}(10) - \mathrm{L}(239) - 4\,\mathrm{L}(515) \qquad \text{(Klingenstierna)}$$
$$\frac{\pi}{4} = 12\,\mathrm{L}(18) + 8\,\mathrm{L}(57) - 5\,\mathrm{L}(239) \qquad \text{(Gauss)}$$
$$\frac{\pi}{4} = 16\,\mathrm{L}(21) + 3\,\mathrm{L}(239) + 4\,\mathrm{L}(1042/3)$$
$$\frac{\pi}{4} = 22\,\mathrm{L}(28) + 2\,\mathrm{L}(443) - 5\,\mathrm{L}(1393) - 10\,\mathrm{L}(11018)$$
$$\frac{\pi}{4} = 22\,\mathrm{L}(38) + 17\,\mathrm{L}(601/7) + 10\,\mathrm{L}(8149/7) \qquad \text{(Sebah)}$$
$$\frac{\pi}{4} = 44\,\mathrm{L}(57) + 7\,\mathrm{L}(239) - 12\,\mathrm{L}(682) + 24\,\mathrm{L}(12943) \qquad \text{(Störmer)}$$
$$\frac{\pi}{4} = 88\,\mathrm{L}(172) + 51\,\mathrm{L}(239) + 32\,\mathrm{L}(682) + 44\,\mathrm{L}(5357) + 68\,\mathrm{L}(12943) \qquad \text{(Störmer)}$$

For example, more than 100 three terms formulae are known and are easy to generate by mean of dedicated algorithms.

5 BBP series

In 1995, Bailey, Borwein and Plouffe (*BBP*) found a new kind of formula which allows to compute directly the d-th digit of π in basis 2 (See [2].)

$$\pi = \sum_{k\geq 0}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)\frac{1}{16^k}.$$

Other such formulae are available.

$$\begin{aligned}
\pi &= \sum_{k\geq 0}\left(\frac{2}{4k+1} + \frac{2}{4k+2} + \frac{1}{4k+3}\right)\frac{(-1)^k}{4^k} \\
\pi &= \sum_{k\geq 0}\left(\frac{2}{8k+1} + \frac{1}{4k+1} + \frac{1}{8k+3} - \frac{1}{16k+10} - \frac{1}{16k+12} - \frac{1}{32k+28}\right)\frac{1}{16^k} \\
\pi &= \frac{1}{64}\sum_{k\geq 0}\left(-\frac{32}{4k+1} - \frac{1}{4k+3} + \frac{256}{10k+1} - \frac{64}{10k+3} - \frac{4}{10k+5} - \frac{4}{10k+7} + \frac{1}{10k+9}\right)\frac{(-1)^k}{1024^k} \\
\pi\sqrt{2} &= \sum_{k\geq 0}\left(\frac{4}{6k+1} + \frac{1}{6k+3} + \frac{1}{6k+5}\right)\frac{(-1)^k}{8^k} \\
\frac{8\pi^2}{9} &= \sum_{k\geq 0}\left(\frac{16}{(6k+1)^2} - \frac{24}{(6k+2)^2} - \frac{8}{(6k+3)^2} - \frac{6}{(6k+4)^2} + \frac{1}{(6k+5)^2}\right)\frac{1}{64^k}
\end{aligned}$$

The series with 1024^k is efficient and due to Fabrice Bellard (1997).

6 Ramanujan's series

Most of those series and many others were found by the Indian prodigy Srinivasa Ramanujan (1887-1920). ([3], [8])

$$\begin{aligned}
\frac{2}{\pi} &= 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \cdots \\
\frac{4}{\pi} &= 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2.4}\right)^2 + \left(\frac{1.3}{2.4.6}\right)^2 + \left(\frac{1.3.5}{2.4.6.8}\right)^2 + \cdots \qquad \text{(Forsyth)} \\
\frac{1}{\pi} &= \sum_{k\geq 0}\binom{2k}{k}^3\frac{(42k+5)}{2^{12k+4}} \\
\frac{1}{\pi} &= \frac{1}{72}\sum_{k\geq 0}(-1)^k\frac{(4k)!}{(k!)^4 4^{4k}}\frac{(23+260k)}{18^{2k}} \\
\frac{1}{\pi} &= \frac{1}{3528}\sum_{k\geq 0}(-1)^k\frac{(4k)!}{(k!)^4 4^{4k}}\frac{(1123+21460k)}{882^{2k}}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\pi} &= \frac{2\sqrt{2}}{9801}\sum_{k\geq 0}\frac{(4k)!}{(k!)^4 4^{4k}}\frac{(1103+26390k)}{99^{4k}} \\
\frac{1}{\pi} &= 12\sum_{k\geq 0}(-1)^k\frac{(6k)!}{(3k)!(k!)^3}\frac{(13591409+545140134k)}{640320^{3k+3/2}} && \text{(Chudnovsky)} \\
\frac{1}{\pi} &= 12\sum_{k\geq 0}(-1)^k\frac{(6k)!}{(3k)!(k!)^3}\frac{(A+Bk)}{C^{3k+3/2}} && \text{(Borwein)}
\end{aligned}$$

In the last formula

$$\begin{aligned}
A &= 1657145277365+212175710912\sqrt{61}, \\
B &= 107578229802750+13773980892672\sqrt{61}, \\
C &= 5280(236674+30303\sqrt{61}),
\end{aligned}$$

and each additional term in the series adds about 31 digits!

7 Other series

$$\begin{aligned}
\frac{\pi-3}{6} &= \sum_{k\geq 1}\frac{(-1)^{k+1}}{36k^2-1} \\
\pi-3 &= \sum_{k\geq 1}\frac{(-1)^{k+1}}{k(k+1)(2k+1)} \\
\frac{\pi^3+8\pi-56}{8} &= \sum_{k\geq 1}\frac{(-1)^{k+1}}{k(k+1)(2k+1)^3} \\
\frac{\pi}{16} &= \sum_{k\text{ odd}}\frac{(-1)^{(k-1)/2}}{k(k^4+4)} && \text{(Glaisher)} \\
\frac{\pi}{4} &= 1-16\sum_{k\geq 0}\frac{1}{(4k+1)^2(4k+3)^2(4k+5)^2} && \text{(Lucas)} \\
10-\pi^2 &= \sum_{k\geq 1}\frac{1}{k^3(k+1)^3} \\
\frac{\pi^2-8}{16} &= \sum_{k\geq 1}\frac{1}{(4k^2-1)^2} && \text{(Euler)} \\
\frac{32-3\pi^2}{64} &= \sum_{k\geq 1}\frac{1}{(4k^2-1)^3} && \text{(Euler)} \\
\frac{\pi^4+30\pi^2-384}{768} &= \sum_{k\geq 1}\frac{1}{(4k^2-1)^4} && \text{(Euler)}
\end{aligned}$$

$$\begin{aligned}
\frac{2\pi\sqrt{3}}{9} &= \sum_{k\geq 0} \frac{k!^2}{(2k+1)!} = 1 + \frac{1}{6} + \frac{1}{30} + \frac{1}{140} + \cdots \\
\frac{2\pi\sqrt{3}}{27} + \frac{4}{3} &= \sum_{k\geq 0} \frac{k!^2}{(2k)!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{20} + \frac{1}{70} + \cdots \\
\frac{\pi\sqrt{3}}{9} &= \sum_{k\geq 1} \frac{k!^2}{(2k)!k} \\
\frac{\pi^2}{18} &= \sum_{k\geq 1} \frac{k!^2}{(2k)!k^2} \qquad \text{(Euler)} \\
\frac{17\pi^4}{3240} &= \sum_{k\geq 1} \frac{k!^2}{(2k)!k^4} \qquad \text{(Comtet)} \\
\frac{\pi}{3} &= \sum_{k\geq 0} \frac{(2k)!}{(2k+1)16^k k!^2} \\
\pi + 3 &= \sum_{k\geq 1} \frac{k!^2 k 2^k}{(2k)!} \\
\pi &= \sum_{k\geq 0} \frac{(25k-3)k!(2k)!}{2^{k-1}(3k)!} \qquad \text{(Gosper 1974)} \\
1 - \frac{8}{\pi^2} &= \sum_{k\geq 0} \frac{(4k+3)}{2^{8k+4}} C_k^4 \qquad \text{(Victor)}
\end{aligned}$$

C_n are Catalan's numbers:

$$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14,\ C_5 = 42,\ \ldots,\ C_n = \frac{(2n)!}{n!^2(n+1)}.$$

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