1 The geometric period

Up to the Seventeenth Century, approximations of π were obtained by mean of geometrical considerations. Most of the methods were dealing with regular polygons circumscribed about and inscribed in the circle. The perimeter or the area of those polygons were calculated with elementary geometrical rules.

During this period the notation π was not used and it was not yet a constant but just a geometrical ratio or even just implicit.

1.1 Ancient estimations

1.1.1 Egypt

In one of the oldest mathematical text, the *Rhind papyrus* (from the name of the Egyptologist Henry Rhind who purchased this document in 1858 at Luxor), the scribe Ahmes copied, around 1650 B.C.E., eighty-five mathematical problems. Among those is given a rule, *the problem 48*, to find the area of a circular field of diameter 9: *take away 1/9 of the diameter and take the square of the remainder.* In modern notation, it becomes

$$A = \left(d - \frac{1}{9}d\right)^2 = \left(\frac{8}{9}\right)^2 d^2,$$

(A is the area of the field and d it's diameter): so if we use the formula $A = \pi d^2/4$, comes the following approximation

$$\pi = 4\left(\frac{8}{9}\right)^2 = \left(\frac{4}{3}\right)^4 \approx 3.1605.$$

This accuracy is astonishing for such ancient time. See [4] for a possible justification of this value.

1.1.2 Babylon

On a Babylonian cuneiform tablet from Susa, about 2000 B.C.E., and discovered in 1936, the ratio of the perimeter of the circle to its diameter was founded to be

$$\pi = 3 + \frac{1}{8} = 3.125,$$

and this estimation is one of the oldest we know.

1.2 Archimedes' method

The famous treatise *On the measurement of the Circle* from the Greek mathematician and engineer Archimedes of Syracuse (287-212 B.C.E.) is a major step

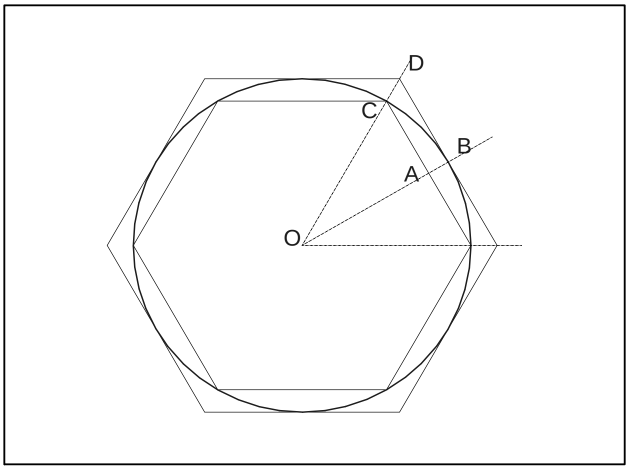

Figure 1: Archimedes's hexagons

in the knowledge of the circle properties. Among those are given the following numerical bounds for π

$$\begin{aligned} 3+\frac{10}{71} &< \pi < 3+\frac{1}{7}=\frac{22}{7} \\ 3.140845 &< \pi < 3.142857, \end{aligned}$$

and, that is a novelty, it's the first algorithm which allows, *in theory*, to compute as many digits of π as required.

The proof uses a recursive algorithm starting with two regular hexagons (see figure (1)), one is circumscribed and the other is inscribed. Archimedes then subdivides both polygons by 2, so the number of sides of the polygons is 6.2^n (respectively $6, 12, 24, 48, 96$ sides in his computations). The perimeters of the last polygons gives the announced bounds.

In modern notations this gives, for the circumscribed polygons, the following recursive sequence (where u_n is half the length of the side of the polygon and v_n the distance from a vertex of the polygon to the center of the circle)

$$\begin{aligned} u_{n+1} &= \frac{u_n}{1+v_n} \\ v_{n+1} &= \sqrt{1+u_{n+1}^2} \\ \pi_n &= 6.2^n u_n \end{aligned}$$

starting with the regular hexagon for which: $u_0 = 1/\sqrt{3}, v_0 = 2/\sqrt{3}, \pi_0 = 2\sqrt{3}$.

We may observe that Archimedes needed to compute square roots to achieve his computation, and in particular he used a good approximation of $\sqrt{3} \approx$

265/153 (this value is one of the partial quotients of the continued fraction of $\sqrt{3}$). The upper bound founded by Archimedes is given with $n = 4$.

Now we observe the value of the first iterates of this sequence:

$$\begin{aligned}
\pi_0 &= 3.(46410161513...) \\
\pi_1 &= 3.(21539030917...) \\
\pi_2 &= 3.1(5965994209...) \\
\pi_3 &= 3.14(608621513...) \\
\pi_4 &= 3.14(271459964...) \\
\pi_{10} &= 3.141592(92738...) \\
\pi_{16} &= 3.141592653(65...)
\end{aligned}$$

The same kind of iteration may be written for the inscribed polygons. Archimedes method is the first known algorithm, in theory, to compute π at a desired accuracy.

For example, π_{16} is a nine digits approximation and it is the perimeter of a polygon of 393216 sides. This approximation was given much later than Archimedes by François Viète (1540-1603) after a long calculation. He gave the bounds [9]:

$$3.1415926535 < \pi < 3.1415926537.$$

1.2.1 Trigonometric formulation

Using trigonometric functions (unknown to Archimedes), it's easy to show that the perimeter of the circumscribed polygon with n sides on a circle of radius r is given by

$$L_n = 2nr \tan\left(\frac{\pi}{n}\right),$$

and the perimeter of the inscribed polygon is

$$l_n = 2nr \sin\left(\frac{\pi}{n}\right).$$

It follows the bounds for π

$$n \sin\left(\frac{\pi}{n}\right) < \pi < n \tan\left(\frac{\pi}{n}\right)$$

and, for example, with $n = 6$ (hexagon)

$$3 < \pi < 2\sqrt{3}.$$

If we observe that, when n is large,

$$\begin{aligned}
n \sin\left(\frac{\pi}{n}\right) &= \pi - \frac{\pi^3}{6n^2} + O\left(\frac{1}{n^4}\right) \\
n \tan\left(\frac{\pi}{n}\right) &= \pi + \frac{\pi^3}{3n^2} + O\left(\frac{1}{n^4}\right),
\end{aligned}$$

this suggests to form (in order to eliminate the term in $1/n^2$)

$$\frac{2}{3}n\sin\left(\frac{\pi}{n}\right)+\frac{1}{3}n\tan\left(\frac{\pi}{n}\right)=\pi+O\left(\frac{1}{n^4}\right)$$

whose convergence is much faster than Archimedes' method. This was noted by a geometric mean, in 1621, by Willebrod Snellius (1580-1626) in his work *Cyclometricus.* For example with the hexagons ($n=6$), it becomes

$$\pi\approx 2+\frac{2}{3}\sqrt{3}=3.1(547...),$$

and with the last polygons considered by Archimedes ($n=96$)

$$\pi\approx 3.141592(833...).$$

See [7] for other accelerations of this method.

1.3 Pfaff formulation

In 1800, Johann Friedrich Pfaff (1765-1825) gave a modern and analytical formulation of the previous geometric iteration. Let a_n and b_n be respectively the length of the circumscribed and inscribed regular polygon with 6.2^n sides, then starting with $a_0=2\sqrt{3}, b_0=3$ and

$$\begin{aligned} a_{n+1} &= \frac{2a_nb_n}{a_n+b_n} \\ b_{n+1} &= \sqrt{a_{n+1}b_n} \end{aligned}$$

we have

$$\begin{aligned} b_n &< b_{n+1}<a_{n+1}<a_n \\ \pi &= \lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n \end{aligned}$$

so that π is the common limit of the two sequences (a_n,b_n), and the error is $O(4^{-n})$.

If we set $\alpha_n=1/u_n$ and $\beta_n=1/v_n$ the iteration becomes (starting with $\alpha_0=1/(2\sqrt{3}), \beta_0=1/3$)

$$\begin{aligned} \alpha_{n+1} &= \frac{\alpha_n+\beta_n}{2} \\ \beta_{n+1} &= \sqrt{\alpha_{n+1}\beta_n} \end{aligned}$$

and

$$\frac{1}{\pi}=\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n.$$

It's interesting to note that this formulation is not very far from the *AGM iteration* which is one of the recent and very efficient methods used to compute many digits of π.

1.4 Record of computation during the geometric period

Here are some other approximations computed by various mathematicians and mostly by mean of polygons:

- 150 : Claudius Ptolemy (c. 85-165, Egypt) published 3.141666... in his *Almagest* (an astronomical treatise), the value was given in sexagesimal fractions $3 + 8/60 + 30/60^2$.
- 263 : Liu Hui (China) gave $3927/1250 = 3.1416$ with a polygon of 3072 sides [6].
- 480? : Zu Chongzhi (429-500, China) determined 355/113 and also the very impressive bounds $3.1415926 < \pi < 3.1415927$ probably by Liu's method [6].
- 499 : Aryabhata (c. 476-550, India) gave 3.1416 maybe by mean of a polygon of 384 sides. In fact, in rule 10 of the *Aryabhatiya* we are told: *add four to one hundred, multiply by eight and then add sixty-two thousand. The result is approximately the circumference of a circle of diameter twenty thousand.*
- 830 : Al'Khwarizmi (c. 780-850, Persia): $22/7, \sqrt{10}$ and 62832/20000. The term *algorithm* came from his name.
- 1220 : Leonardo of Pisa (1180-1240, Italy): 3.141818. He is better known as *Fibonacci*.
- 1424 : Al-Kashi (c. 1380-1429, Samarkand): 14 digits with a polygon of 6.2^{27}sides in his *Treatise on the Circumference.*
- 1579 : François Viète (1540-1603, France): 9 digits with a polygon of 393216 sides.
- 1593 : Adrianus Romanus (1561-1615, Netherlands): 15 digits with a polygon of 2^{30} sides.
- 1596 : Ludolph van Ceulen (1540-1610, Germany): 20 digits with a polygon of 60.2^{33} sides.
- 1609 : Ludolph van Ceulen: 35 digits with a polygon of 2^{62} sides. He spent a considerable part of his life with such computations. π was often known as the *Ludolphine Number* in Germany and his digits are engraved on his tombstone.
- 1621 : Willebrod Snellius (1580-1626, Netherlands): 34 digits with a polygon of only 2^{30} sides thanks to an acceleration of Archimedes' method.

A very complete outline of the history of π is given in [8] and a more modern one are in [2] and [3].

References

[1] Le Petit Archimède, no. hors série, *Le nombre* π, (1980)

[2] J. Arndt and C. Haenel, $\pi-$ *Unleashed*, Springer, (2001)

[3] J.P. Delahaye, *Le fascinant nombre* π, Bibliothèque Pour la Science, Belin, (1997)

[4] H. Engels, *Quadrature of the Circle in Ancient Egypt*, Historia Mathematica, (1977), vol. 4, p. 137-140

[5] T.L. Heath, *The Works of Archimedes*, Cambridge University Press, (1897)

[6] L.Y. Lam and T.S. Ang, *Circle Measurements in Ancient China*, Historia Mathematica, (1986), vol. 13, p. 325-340

[7] G.M. Phillips, *Archimedes the Numerical Analyst*, The American Mathematical Monthly, (1981), vol. 88, p. 165-169

[8] H.C. Schepler, *The Chronology of Pi*, Mathematics Magazine, (1950)

[9] F. Viète, *Opera Mathematica* (reprinted), Georg Olms Verlag, Hildesheim, New York, (1970)