Pythagoras' Constant : $\sqrt{2}$

Pascal Sebah and Xavier Gourdon
numbers.computation.free.fr/Constants/constants.html

November 15, 2001

Abstract

An introduction to the probably first irrational known constant $\sqrt{2}$ and to its computation using elementary methods or high order algorithms.

$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769 4...$

There are certainly people who regard $\sqrt{2}$ as something perfectly obvious but jib at $\sqrt{-1}$. This is because they think they can visualise the former as something in physical space but not the latter. Actually $\sqrt{-1}$ is a much simpler concept.

Edward Charles Titchmarsh (1899-1963).

1 Introduction

The constant $\sqrt{2}$ is famous because it's probably one of the first irrational numbers discovered. According to the Greek philosopher Aristotle (384-322 BC), it was the Pythagoreans around 430 BC who first demonstrated the irrationality of the diagonal of the unit square and this discover was terrible for them because all their system was based on integers and fractions of integers. Later, about 2300 years ago, in Book X of the impressive *Elements,* Euclid (325-265 BC) showed the irrationality of every nonsquare integer (consult [7] for an introduction to early Greek Mathematics). This number was also studied by the ancient Babylonians. The history of the famous sign $\surd$ goes back up to 1525 in a treatise named *Coss* where the German mathematician Christoff Rudolff (1499-1545) used a similar sign to represent square roots.

Definition 1 *$\sqrt{2}$ is the unique positive root of the polynomial equation :*

$$x^2 - 2 = 0.$$

Theorem 2 *$\sqrt{2}$ is an irrational and algebraic number.*

Proof. *Suppose that $\sqrt{2} = p/q$ where p and q are relatively primes, then $p^2 = 2q^2$ therefore p is even and $p = 2p'$ which leads to $q^2 = 2p'^2$ and to the fact that $q = 2q'$. This is in contradiction with p and q being relatively primes.* ■

We will now introduce some of the techniques available to compute this number.

2 Sequences with integers

2.1 Elementary approach

An elementary and ancient algorithm consists in using the double sequence

$$\left\{ \begin{array}{l} p_{n+1} = p_n + 2q_n \\ q_{n+1} = p_n + q_n \end{array} \right. \tag{1}$$

which verifies

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n/q_n + 2}{p_n/q_n + 1}$$

so that :

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{p_n}{q_n} = \sqrt{2}.$$

It is of interest to note that this quite simple recursion only uses additions with integers and eventually a final division to convert the fraction into the usual decimal representation. Denoting the error by $\varepsilon_n = |\sqrt{2} - p_n/q_n|$, one easily see that $\varepsilon_{n+1} < \varepsilon_n/5$, which involves a *geometrical convergence* (that is, one digit will be added at every N iterations).

Starting with $p_0 = q_0 = 1$ we obtain :

$$\left| \begin{array}{l} x_1 = 1.(500\ldots), \\ x_3 = 1.41(666...), \\ x_6 = 1.4142(011...), \\ x_9 = 1.414213(624...), \end{array} \right.$$

$x_3 = 17/12$ was used by Mesopotamians to replace $\sqrt{2}$. It may be convenient to use a matrix representation of the sequence, we write the sequence like this :

$$\left(\begin{array}{c} p_{n+1} \\ q_{n+1} \end{array} \right) = \left(\begin{array}{cc} 1 & 2 \\ 1 & 1 \end{array} \right) \left(\begin{array}{c} p_n \\ q_n \end{array} \right)$$

so that

$$\left(\begin{array}{c} p_{n+1} \\ q_{n+1} \end{array} \right) = A^n \left(\begin{array}{c} p_1 \\ q_1 \end{array} \right) = \left(\begin{array}{cc} 1 & 2 \\ 1 & 1 \end{array} \right)^n \left(\begin{array}{c} p_1 \\ q_1 \end{array} \right).$$

2.1.1 Quadratic convergence

This matrix representation may be used with $n = 2^p$ and thanks to the relation (exercise : show this by recurrence) :

$$A^n = \left(\begin{array}{cc} a_n & 2b_n \\ b_n & a_n \end{array} \right)$$

comes the following algorithm

$$\left\{ \begin{array}{l} a_{p+1} = a_p^2 + 2b_p^2 \\ b_{p+1} = 2a_p b_p \end{array} \right.$$

starting with $a_0 = b_0 = 1$. It's easy to see that this new process has a *quadratic convergence* rate and its first iterates are given here :

$$(3,2),(17,12),(577,408),(665857,470832),(886731088897,627013566048),...$$

and

$$\left|\frac{886731088897}{627013566048}-\sqrt{2}\right|<10^{-24}.$$

Note that, in each step of this algorithm, the product of large integers is required and this may be done very efficiently using fast multiplication methods. This method is related to Newton's iteration by observing that :

$$x_{p+1}=\frac{a_{p+1}}{b_{p+1}}=\frac{a_p^2+2b_p^2}{2a_pb_p}=\frac{1}{2}\frac{a_p}{b_p}+\frac{b_p}{a_p}=\frac{x_p}{2}+\frac{1}{x_p},$$

which will be derived by different means later in this essay (see section 6).

2.1.2 Cubic convergence and quintic convergence

The same approach leads to the *Cubic* sequence

$$\begin{cases} a_{p+1}=a_p\left(a_p^2+6b_p^2\right) \\ b_{p+1}=b_p\left(3a_p^2+2b_p^2\right) \end{cases}$$

starting with $a_0 = b_0 = 1$ the first iterates are given by :

$$(7,5),(1393,985),(10812186007,7645370045),...$$

and also to the *Quintic* sequence

$$\begin{cases} a_{p+1}=a_p\left(a_p^4+20a_p^2b_p^2+20b_p^4\right) \\ b_{p+1}=b_p\left(5a_p^4+20a_p^2b_p^2+4b_p^4\right) \end{cases}$$

giving the iterates

$$(41,29),(1855077841,1311738121),...$$

We have generated algorithms of this nature at any order of convergence using in the sequence polynomials of higher degrees.

2.2 Modification of the sequence

If we set $u_n = p_n + q_n$ and $v_n = q_n$ then the double sequence (1) is equivalent to this new one :

$$\begin{cases} u_{n+1}=2u_n+u_{n-1} \\ v_{n+1}=u_n \end{cases}$$

and $u_n/v_n = u_n/u_{n-1} = (p_n + q_n)/q_n$ converge to $\sqrt{2}+1$. So starting with $u_0 = 1, u_1 = 2$ gives the set of u_n :

$$(1,2,5,12,29,70,169,408,985,2378,5741,13860,33461,80782,195025,470832,...)$$

those numbers are sometime called *Pell numbers* and the sequence u_n is a *Pell sequence*. The first 41 numbers may be found in [10] as well as other similar sequences like *Lucas sequences* used in Primality tests. For example

$$\frac{u_{15}}{u_{14}} - 1 = \frac{470832}{195025} - 1 = 1.4142135623(637...)$$

and it was found with very little efforts.

This method is probably due to ancient Babylonians.

2.3 Improvement of the sequence

It's possible to improve this method by using a more general sequence :

$$\begin{cases} p_{n+1} = (A+B)p_n + 2Aq_n \\ q_{n+1} = 2Bp_n + (A+B)q_n \end{cases}$$

Easy manipulations show that

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \sqrt{\frac{A}{B}}$$

and that for the error

$$\varepsilon_n = \left| \sqrt{\frac{A}{B}} - x_n \right|,$$

we have the bound

$$\varepsilon_{n+1} < \varepsilon_n \left(\sqrt{\frac{A}{B}} - 1 \right)^2.$$

The more A/B is close to 1 the more the speed of convergence is increased.

Example 3 *To illustrate the method let's use the relation*

$$\sqrt{2} = \frac{7}{5}\sqrt{\frac{50}{49}}$$

which corresponds to $A = 50$ *and* $B = 49$ *and the iterative system becomes :*

$$p_1 = q_1 = 1, \qquad \begin{cases} p_{n+1} = 99p_n + 100q_n \\ q_{n+1} = 98p_n + 99q_n \end{cases}$$

as for the error, we have

$$\varepsilon_{n+1} < \varepsilon_n \left(\sqrt{1 + \frac{1}{49}} - 1 \right)^2 < \varepsilon_n / 9604.$$

Here are the first iterates :

$$\left| \begin{array}{l} x_1 = 1.4, \\ x_2 = 1.414213(197...), \\ x_3 = 1.4142135623(637...), \\ x_4 = 1.41421356237309(481...) \end{array} \right.$$

and almost 4 new digits are added with each step.

3 Continued fraction

3.1 Elementary continued fraction

The idea is to write $\sqrt{2} = 1 + 1/\alpha_1$ with $\alpha_1 = 2.4142135... = 2 + 1/\alpha_2$ and so on... This gives the well-known development :

Theorem 4 *(Bombelli 1572, [4]).*

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+...}}$$

It is more convenient to use the notation

$$\sqrt{2} = [1; 2, 2, 2, 2, ...].$$

The development is periodic of period 1 with number $\{2\}$. It's a general result : square roots can always be represented with a periodic continued fraction development [5]. It's possible to show that the previous sequence also gives the continued development of $\sqrt{2}$. We give the first rational approximations :

$$\left(\frac{p_k}{q_k}\right)_{k=1...} = \left(\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \frac{1393}{985}, \frac{3363}{2378}, \frac{8119}{5741}, \frac{19601}{13860}, \frac{47321}{33461}, \cdots\right).$$

3.2 Other continued fractions

Faster converging continued fractions may be obtained, for example :

$$5\sqrt{2} - 7 = [0; 14, 14, 14, 14, ...].$$

The development is periodic of period 1 with number $\{14\}$. This time, the first rational approximations are :

$$\left(\frac{p_k}{q_k}\right)_{k=1...} = \left(\frac{1}{14}, \frac{14}{197}, \frac{197}{2772}, \frac{2772}{39005}, \frac{39005}{548842}, \frac{548842}{7722793}, \frac{7722793}{108667944}, \cdots\right)$$

and of course $\sqrt{2} \approx 7/5 + p_k/(5q_k)$, the error is about $1/q_k^2$. Observe that here $p_k = q_{k-1}$ which simplifies the evaluation of the approximations.

Other fast converging continued fractions are given by

$$29\sqrt{2} - 41 = [0; 82, 82, 82, 82, ...]$$

and

$$169\sqrt{2} - 239 = [0; 478, 478, 478, 478, ...].$$

It's possible to find interesting continued fractions for numbers of the form $m\sqrt{2} - n$ and it's not hard to show that if the integers m and n are such as

$$n^2 - 2m^2 = -1$$

(this is a nonlinear Diophantine equation and it's called a *Pell's equation*) then

$$m\sqrt{2}-n=[0;2n,2n,2n,2n,...].$$

From a fundamental result on Pell's equations the solutions are contained in the convergents of the simple continued fraction of $\sqrt{2}$ (there are exactly the convergents of even rank [5] in the case of $\sqrt{2}$). It's also possible to show that there are given by the sequence

$$\begin{cases} m_{k+1}=3m_k+2n_k \\ n_{k+1}=4m_k+3n_k \end{cases}$$

starting with $(1,1)$.

Hence the first couples (m_k, n_k) are

$$(1,1),(5,7),(29,41),(169,239),(985,1393),(5741,8119),(33461,47321),...$$

Example 5 *With the couple* $(33461, 47321)$ *we have the approximation*

$$33461\sqrt{2}-47321\approx\frac{1}{2.47321}$$

that is

$$\sqrt{2}\approx\frac{47321}{33461}+\frac{1}{3166815962}=1.4142135623730950488(369...)$$

(19 correct digits).

4 Infinite products

The two well-known Infinite products for $\cos x$ and $\sin x$ are :

Theorem 6 *(Euler 1748, [2])*

$$\begin{aligned} \cos x &= \left(1-\tfrac{4x^2}{\pi^2}\right)\left(1-\tfrac{4x^2}{9\pi^2}\right)\left(1-\tfrac{4x^2}{25\pi^2}\right)\cdots \\ \sin x &= x\left(1-\tfrac{x^2}{\pi^2}\right)\left(1-\tfrac{x^2}{4\pi^2}\right)\left(1-\tfrac{x^2}{9\pi^2}\right)\cdots \end{aligned}$$

Setting $x=\pi/4$ in the cosine product leads to

$$\frac{1}{\sqrt{2}}=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{36}\right)\left(1-\frac{1}{100}\right)...$$

therefore

$$\sqrt{2}=\left(\frac{2.2}{1.3}\right)\left(\frac{6.6}{5.7}\right)\left(\frac{10.10}{9.11}\right)\left(\frac{14.14}{13.15}\right)...=\prod_{k\geq 0}\frac{(4k+2)^2}{(4k+1)(4k+3)}, \qquad (2)$$

or, which is equivalent, to the aesthetic formula :

$$\sqrt{2}=\prod_{k\geq 0}\left(1+\frac{1}{4k+1}\right)\left(1-\frac{1}{4k+3}\right)=\left(1+\frac{1}{1}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)... \tag{3}$$

The convergence is extremely slow as we may observe with some iterates :

$$\left|\begin{array}{l} x_1 = 1.(333...), \\ x_{10} = 1.4(054...), \\ x_{100} = 1.41(332...), \\ x_{1000} = 1.414(125...). \end{array}\right.$$

Euler gave the interesting product (2) in 1748 [2], and it's very similar to John Wallis formula for π :

$$\frac{\pi}{4}=\left(\frac{2.4}{3.3}\right)\left(\frac{4.6}{5.5}\right)\left(\frac{6.8}{7.7}\right)\left(\frac{8.10}{9.9}\right)...,$$

while the product (3) can be compared to the celebrated series

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...$$

With $x=\pi/8$ in the infinite product we extract the also slow converging infinite product

$$2+\sqrt{2}=4.\left(\frac{3.5}{4.4}\right)^2\left(\frac{11.13}{12.12}\right)^2\left(\frac{19.21}{20.20}\right)^2\left(\frac{27.29}{28.28}\right)^2...$$

5 Taylor's expansions

From the Taylor expansion of $(1+x)^{1/2}$ or $(1-x)^{-1/2}$, it's possible to find another class of algorithms based on series computation.

Theorem 7 *(Newton 1665). Let $-1<x<1$, then*

$$\begin{array}{c} \sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{2.4}x^2+\frac{1.3}{2.4.6}x^3-\cdots \\ 1/\sqrt{1-x}=1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\cdots \end{array} \tag{4}$$

This theorem was first stated by Isaac Newton (1643-1727) and a correct proof appears later and is due to Leonhard Euler (1707-1783).

Applying the first expansion in (4) which is still valid for $x=1$ produces the very slowly convergent and alternating series

$$\sqrt{2}=1+\frac{1}{2}-\frac{1}{2.4}+\frac{1.3}{2.4.6}-\frac{1.3.5}{2.4.6.8}+...,$$

which first terms are :

$$\left| \begin{array}{l} x_1 = 1.(500...), \\ x_2 = 1.(375...), \\ x_3 = 1.4(375...), \\ x_4 = 1.(398...). \end{array} \right.$$

A nice improvement is to use the previous results on continuous fractions. We can write :

$$\sqrt{2} = \frac{p_k}{q_k}\sqrt{\frac{2q_k^2}{p_k^2}},$$

and for each value of k, this will provide a formula where the number inside the square root of the right hand side of the formula is near 1. When k increases this number tends to 1. Using this remark leads to a sequence of formulae for $\sqrt{2}$, which are more and more efficient :

$$\sqrt{2} = \left(\frac{3}{2}\sqrt{\frac{8}{9}}, \frac{7}{5}\sqrt{\frac{50}{49}}, \frac{17}{12}\sqrt{\frac{288}{289}}, \frac{41}{29}\sqrt{\frac{1682}{1681}}, \frac{99}{70}\sqrt{\frac{9800}{9801}}, \frac{239}{169}\sqrt{\frac{57122}{57121}}, \cdots \right).$$

Example 8 *Using this last formula jointly with Newton's series expansion gives the very fast converging and easy to implement sequence*

$$\sqrt{2} = \frac{239}{169}\left(1 + \frac{1}{2}\frac{1}{57121} - \frac{1}{8}\frac{1}{57121^2} + \frac{3}{48}\frac{1}{57121^3} - \cdots \right)$$

for which the first terms are :

$$\left| \begin{array}{l} x_1 = 1.4142(011...), \\ x_2 = 1.414213562(427...), \\ x_3 = 1.41421356237309(457...), \\ x_4 = 1.41421356237309504880(687...). \end{array} \right.$$

With a very little calculation we have computed 20 digits of $\sqrt{2}$.

Example 9 *(Euler 1755, [4], [8]). The relation* $\sqrt{2} = (7/5)/\sqrt{49/50}$ *produces the nice and easy to compute by hand series expansion :*

$$\sqrt{2} = \frac{7}{5}\left(1 + \frac{1}{100} + \frac{1.3}{100.200} + \frac{1.3.5}{100.200.300} + ...\right).$$

The main advantage of using Taylor's formulae is to require only basic multi-precision operations between numbers.

You can download a small clear `C` program which uses this type of formula with a classical algorithm to compute $\sqrt{2}$: see the *Easy programs for constants page* at [3].

6 Newton's iteration

A very efficient way to compute square roots is to use the *Newton iteration* [9] on the function $f(x) = x^2 - 2$. It gives the following sequence :

$$x_0 = 1,$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{x_n}{2} + \frac{1}{x_n},$$

x_{n+1} can also be interpreted as the simple mean between the approximation x_n and $2/x_n$ which is also another approximation.

The first terms of this sequence are :

$$\left| \begin{array}{l} x_1 = 1.(500...), \\ x_2 = 1.41(666...), \\ x_3 = 1.41421(568...), \\ x_4 = 1.41421356237(468...). \end{array} \right.$$

Number D of correct digits after n iterations with the elementary Newton iteration :

n	1	2	3	4	5	6	7	8	9	10
D	0	2	5	11	24	48	97	195	391	783

It's a well-known result that the rate of convergence of Newton's method and therefore of this sequence is *quadratic* (the number of digits is doubling at each iteration) for a starting point x_0 sufficiently close to the root of the equation.

Computing $\sqrt{2}$ with this algorithm is convenient if one can compute easily the inverse of x_n. In fact, it is simpler to use only multiplications. This can be reached by using a modified sequence which converges to $1/\sqrt{2}$: it consists in using the Newton iteration but this time with the function $f(x) = 1/x^2 - 2$, yielding the sequence

$$x_0 = \frac{1}{2}, \qquad x_{n+1} = x_n\left(\frac{3}{2} - x_n^2\right) = x_n + x_n\left(\frac{1}{2} - x_n^2\right).$$

Observe that in the right hand side relation the increment tends to zero. We give the first iterates :

$$\left| \begin{array}{l} x_1 = 0.(625...), \\ x_2 = 0.(693...), \\ x_3 = 0.70(670...), \\ x_4 = 0.707106(444...), \\ x_5 = 0.707106781186(307...), \end{array} \right.$$

again the convergence is quadratic. A final multiplication by 2 will give the value of $\sqrt{2}$. The advantage of this method is to avoid a multi-precision division at each step and replace it by two multi-precision multiplications. This algorithm is extremely efficient and may be used to compute $\sqrt{2}$ up to billion's of digits.

7 Cubic iteration

7.1 Halley's iteration

Using the second derivative of $f(x) = x^2 - 2$, it's possible to find a sequence with cubic convergence (the number of digits is multiplied by 3 at each step). The general formula is given by *Halley's iteration* (the original form was introduced by the astronomer Edmund Halley (1656-1742) in 1694) :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 - \frac{f(x_n)f''(x_n)}{2f'(x_n)^2}\right)^{-1}.$$

Applying this formula with function $f(x)$ gives :

$$x_0 = 1, \qquad x_{n+1} = x_n\frac{x_n^2+6}{3x_n^2+2} = x_n\left(1 + 2\frac{(2-x_n^2)}{3x_n^2+2}\right).$$

The first terms of this sequence are :

$$\left|\begin{array}{l} x_1 = 1.4(000...), \\ x_2 = 1.414213(197...), \\ x_3 = 1.414213562373095048(795...). \end{array}\right.$$

Number D of correct digits after n iterations with Halley's iteration :

n	1	2	3	4	5	6	7	8	9	10
D	1	6	20	61	185	557	1673	5022	15067	45204

7.2 Another cubic iteration

In [1], an interesting cubic iteration is given which converge to $1/\sqrt{A}$

$$x_{n+1} = \frac{x_n}{8}(15 - 10Ax_n^2 + 3A^2x_n^4).$$

It may be deduced from the general *Householder's iteration* [6]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{f(x_n)f''(x_n)}{2f'(x_n)^2}\right),$$

which has *cubical* convergence for a close enough starting point x_0 (observe the similarity with Halley's iteration).

By applying this general pattern for $f(x) = 1/x^2 - A$ we obtain :

$$x_{n+1} = x_n + \frac{x_n}{8}\left(7 - 10Ax_n^2 + 3A^2x_n^4\right) = x_n + \frac{3A^2x_n}{8}\left(x_n^2 - \frac{1}{A}\right)\left(x_n^2 - \frac{7}{3A}\right).$$

If we explicit this formula with $A = 2$, and starting with $x_0 = 1/2$,

$$x_{n+1} = x_n + \frac{x_n}{8}(7 - 20x_n^2 + 12x_n^4) = x_n + \frac{x_n}{8}(2x_n^2 - 1)(6x_n^2 - 7),$$

and, in this iteration, no multiprecision division is required. The first iterates are :

$$\left| \begin{array}{l} x_1 = 0.(671...), \\ x_2 = 0.70(689...), \\ x_3 = 0.7071067811(398...). \end{array} \right.$$

Number D of correct digits after n iterations with Householder's cubic algorithm :

n	1	2	3	4	5	6	7	8	9	10
D	0	2	10	30	90	269	808	2425	7276	21829

This method may also be very efficient for high precision computation.

8 Quartic and high order iterations

The direct application of the *modified iterations* (see the *Newton's Iteration* pages at [3]) on the function $f(x) = 1/x^2 - 1/2$ produces a set of high order algorithms. It's interesting to note that relatively easy algorithms of any order may be derived.

In the following lines we will use the notation

$$h_n = 1 - \frac{x_n^2}{2}.$$

8.1 Quartic iteration

The *quartic modified iteration* is

$$x_{n+1} = x_n + \frac{x_n}{16}\left(8h_n + 6h_n^2 + 5h_n^3\right).$$

For example, if we take the initial value $x_0 = 3/2$, the first two iterations are

$$\left| \begin{array}{l} x_1 = 1.414(123...), \\ x_2 = 1.41421356237309(494...). \end{array} \right.$$

Number D of correct digits after n iterations with the quartic algorithm :

n	1	2	3	4	5	6	7
D	3	14	63	254	1019	4078	16312

8.2 Quintic iteration

The same method also produces the *quintic (fifth-order) modified iteration*

$$x_{n+1} = x_n + \frac{x_n}{128}\left(64h_n + h_n^2\left(48 + 40h_n + 35h_n^2\right)\right),$$

and again with the initial value $x_0 = 3/2$, the first two iterations are :

$$\left| \begin{array}{l} x_1 = 1.4142(236...), \\ x_2 = 1.4142135623730950488016 88(932...). \end{array} \right.$$

Number D of correct digits after n iterations with the quintic algorithm :

n	1	2	3	4	5	6
D	4	24	123	615	3076	15380

Algorithms with any order of convergence may also be generated easily (for example : sextic, septic, octic, ... iterations are available !) and we end this section with the *octic (eighth-order) iteration* :

$$\left\{ \begin{array}{l} \delta_n = 1024h_n + h_n^2 \left(768 + 640h_n + 560h_n^2 + h_n^2(504h_n + h_n^2 (462 + 429h_n))\right), \\ x_{n+1} = x_n + \frac{1}{2048} x_n \delta_n. \end{array} \right.$$

9 Double Iteration

9.1 Quadratic double iterative procedure

During the first years of computer time (EDSAC group at Cambridge University 1951), the following double sequence (easily deduced from Newton's iteration on the inverse of the square root) was proposed to compute square roots :

$$x_0 = A, h_0 = A - 1,$$

and

$$\left\{ \begin{array}{l} x_{n+1} = x_n \left(1 - \frac{1}{2}h_n\right) \\ h_{n+1} = \frac{1}{4}h_n^2 (h_n - 3) \end{array} \right.$$

then the sequence h_n tends to 0 and the sequence x_n tends to $\sqrt{A}$ $(A < 3)$.

Example 10 *Let* $A = 2$, *therefore*

$$\left| \begin{array}{l} x_1 = 1.(000...), h_1 = -0.5 \\ x_2 = 1.(250...), h_2 = -0.21875... \\ x_3 = 1.(386...), h_3 = -0.03850... \\ x_4 = 1.41(341...), h_4 = -0.001126... \\ x_5 = 1.41421(288...), h_5 = -0.000000951... \end{array} \right.$$

and then the convergence becomes quadratic. The number of non quadratic cycles is reduced when A *tends to* 1. *For example, using* $\sqrt{2} = 7/5\sqrt{50/49}$ *so that* $A = 50/49$, *will lead to a quadratic converge at once.*

9.2 Other rates of convergence

Small modifications in the previous procedure may increase the order of convergence, for example the following algorithm has cubic convergence :

$$x_0 = A, h_0 = A - 1,$$

and

$$\begin{cases} x_{n+1} = x_n \left(1 - \frac{1}{2}h_n + \frac{3}{8}h_n^2\right) \\ h_{n+1} = \frac{1}{64}h_n^3 \left(40 - 15h_n + 9h_n^2\right). \end{cases}$$

Algorithms at any order of convergence may also be generated.

References

[1] J.M. Borwein and P.B. Borwein, *Pi and the AGM - A study in Analytic Number Theory and Computational Complexity*, A Wiley-Interscience Publication, New York, (1987)

[2] L. Euler, *Introduction à l'analyse infinitésimale* (french traduction by Labey), Barrois, ainé, Librairie, (original 1748, traduction 1796), vol. 1, p. 89-90

[3] X. Gourdon and P. Sebah, *Numbers, Constants and Computation*, Internet site at *http://numbers.computation.free.fr/Constants/constants.html.*

[4] E. Hairer and G. Wanner, *L'analyse au fil de l'histoire*, Bibliothèque Scopos, Springer, (2000)

[5] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Science Publications, (1979)

[6] A.S. Householder, *The Numerical Treatment of a Single Nonlinear Equation*, McGraw-Hill, New York, (1970)

[7] V.J. Katz, *A History of Mathematics-An Introduction*, Addison-Wesley, (1998)

[8] K. Knopp, *Theory and application of infinite series*, Blackie & Son, London, (1951)

[9] I. Newton, *Methodus fluxionum et serierum infinitarum*, (1664-1671)

[10] P. Ribenboim, *The new Book of Prime Number Records*, Springer, (1996)